RESTRICTING FOURIER TRANSFORMS OF MEASURES TO CURVES IN $\mathbb{R}^2$

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Abstract. We establish estimates for restrictions to certain curves in $\mathbb{R}^2$ of the Fourier transforms of some fractal measures.

1. Introduction

The starting point for this note was the following observation: if μ is a compactly supported nonnegative Borel measure on $\mathbb{R}^2$ which, for some $\alpha > 3/2$, is α-dimensional in the sense that

$$\mu\big(B(y,r)\big) \lesssim r^{\alpha} \tag{1.1}$$

for $y \in \mathbb{R}^2$ and $r > 0$, then

$$\int_0^\infty |\widehat{\mu}(t,t^2)|^2 \, dt < \infty. \tag{1.2}$$

The proof is easy: writing $d\lambda$ for the measure given by dt on the curve (t,t^2), we see that

$$\int_0^\infty |\widehat{\mu}(t,t^2)|^2 \, dt = \iiint e^{-2\pi i (t,t^2)\cdot(x-y)} \, d\mu(x)\, d\mu(y)\, dt = \iint \widehat{\lambda}(x-y)\, d\mu(x)\, d\mu(y) \lesssim \iint |x_2 - y_2|^{-1/2}\, d\mu(x)\, d\mu(y), \tag{1.3}$$

where we put $x = (x_1, x_2)$ if $x \in \mathbb{R}^2$ and the inequality comes from the van der Corput estimate $|\widehat{\lambda}(x)| \lesssim |x_2|^{-1/2}$. For fixed y, the compact support of μ implies that

$$\int |x_2 - y_2|^{-1/2}\, d\mu(x) \lesssim \sum_{j=0}^{\infty} 2^{j/2} \mu(\{x : |x_2 - y_2| \le 2^{-j}\}) \lesssim \sum_{j=0}^{\infty} 2^{j/2} 2^{j} 2^{-j\alpha}$$

since $\{x : |x_2 - y_2| \le 2^{-j}\}$ can be covered by $\lesssim 2^j$ balls of radius 2^{-j}. Clearly the last sum is finite if $\alpha > 3/2$, and then (1.3) is finite since μ is a finite measure.

Date: September, 2010.

1991 *Mathematics Subject Classification.* 42B10, 28A12.

Key words and phrases. Fourier transforms of fractal measures, Fourier restriction.

The first author is partially supported by the NSF grant DMS-0900865.

The simplemindedness of this argument made it seem unlikely that the index 3/2 is best possible, and the search for that best index was the motivation for this work. Our results here are the following theorems:

Theorem 1.1. *Suppose* $\phi \in C^2([1,2])$ *satisfies the estimates*

$$\phi' \approx m,\ \phi'' \approx m \tag{1.4}$$

for some $m \geq 1$, *and let* $\gamma(t) = \big(t, \phi(t)\big)$. *Suppose* μ *is a nonnegative and compactly supported measure on* $\mathbb{R}^2$ *which is* α*-dimensional in the sense that* (1.1) *holds. Then, for* $\epsilon > 0$,

$$\int_1^2 |\widehat{\mu}\big(R\,\gamma(t)\big)|^2 dt \lesssim R^{-\alpha/2+\epsilon}\, m^{1-\alpha}, \tag{1.5}$$

when $R \geq 2$. *Here the implied constant in* (1.5) *depends only on* α, ϵ, *the implied constants in* (1.1) *and* (1.4), *and the diameter of the support of* μ.

Theorem 1.2. *Suppose* μ *is as in Theorem 1.1,* $p > 1$, *and*
(i) $-1 < \gamma < \alpha p - \alpha/2 - p$ *if* $1 < \alpha < 2$,
(ii) $-1 < \gamma < -1/2$ *if* $1/2 < \alpha \leq 1$,
(iii) $-1 < \gamma < \alpha - 1$ *if* $0 < \alpha \leq 1/2$.
Then

$$\int_0^\infty |\widehat{\mu}(t, t^p)|^2\, t^\gamma\, dt \leq C < \infty, \tag{1.6}$$

where C *depends only on p, the implied constant in* (1.1), *and the diameter of the support of* μ.

Theorem 1.3. *If* (1.6) *holds for* $p > 1$ *and* $\alpha \in (0,2)$ *with* C *as stated in Theorem 1.2, then*
(i) $-1 < \gamma \leq \alpha p - \alpha/2 - p$ *if* $1 < \alpha < 2$,
(ii) $-1 < \gamma \leq -1/2$ *if* $1/2 < \alpha \leq 1$,
(iii) $-1 < \gamma \leq \alpha - 1$ *if* $0 < \alpha \leq 1/2$.

Here are some comments:
(a) Theorem 1.1 is a generalization of Theorem 1 in [7], which was reproved with a simpler argument in [1]. As described in §2, the proof of Theorem 1.1 is just an adaptation of ideas from [7] and [1].
(b) The examples which comprise the proof of Theorem 1.3 are similar in spirit to those in the proof of Proposition 3.2 in [7].
(c) If α_0 is the infimum of the α's for which (1.1) implies (1.2) whenever μ is compactly supported, it follows from Theorem 1.2 that $\alpha_0 \leq 4/3$. Then the proof of Theorem 1.3 and a uniform boundedness argument together imply that $\alpha_0 = 4/3$.
(d) Analogs of Theorem 1.1 have been studied for hypersurfaces in $\mathbb{R}^d$ and, particularly, for the sphere S^{d-1}. See, for example, [3], [4], [5], [6], [1], and [2].

The remainder of this note is organized as follows: the proof of Theorem 1.1 is in §2 and the proofs of Theorems 1.2 and 1.3 are in §3.

2. Proof of Theorem 1.1

As mentioned above, the proof is an adaptation of ideas from [7] and [1]. Specifically, with μ as in Theorem 1.1 and

$$\Gamma_R = \{R\,\gamma(t) : 1 \le t \le 2\},\ \Gamma_{R,\delta} = \Gamma_R + B(0, R^\delta)$$

for $R \ge 2$ and $\delta > 0$, we will modify an uncertainty principle argument from [7] to show that (1.5) follows from the estimate

$$\int_{\Gamma_{R,\delta}} |\widehat{\mu}(y)|^2\, dy \lesssim R^{1-\alpha/2+2\delta}\, m^{2-\alpha}. \tag{2.1}$$

We will then adapt a bilinear argument from [1] to prove (2.1).

So, arguing as in [7], if $\kappa \in C_c^\infty(\mathbb{R}^2)$ is equal to 1 on the support of μ, then

$$\int_1^2 |\widehat{\mu}\big(R\,\gamma(t)\big)|^2 dt = \int_1^2 \Big| \int \widehat{\kappa}\big(R\,\gamma(t) - y\big)\, \widehat{\mu}(y)\, dy \Big|^2 dt \lesssim \int \int_1^2 \big|\widehat{\kappa}\big(R\,\gamma(t) - y\big)\big| dt\ |\widehat{\mu}(y)|^2\, dy. \tag{2.2}$$

If $y = (y_1, y_2)$, then

$$\int_1^2 \big|\widehat{\kappa}(R\,\gamma(t) - y)\big| dt \lesssim \int_1^2 \frac{1}{\big(1 + |R\,\gamma(t) - y|\big)^{10}}\, dt \lesssim \frac{1}{\big(1 + \mathrm{dist}(\Gamma_R, y)\,\big)^8} \int_1^2 \frac{1}{\big(1 + |R\,\phi(t) - y_2|\big)^2}\, dt.$$

Estimating the last integral using the hypothesized lower bound on ϕ', we see from (2.2) that

$$\int_1^2 |\widehat{\mu}\big(R\,\gamma(t)\big)|^2 dt \lesssim \frac{1}{Rm} \int \frac{|\widehat{\mu}(y)|^2}{\big(1 + \mathrm{dist}(\Gamma_R, y)\,\big)^8}\, dy. \tag{2.3}$$

Now

$$\int \frac{|\widehat{\mu}(y)|^2}{\big(1 + \mathrm{dist}(\Gamma_R, y)\,\big)^8}\, dy = \int_{\Gamma_{R,\epsilon/2}} + \sum_{j=2}^\infty \int_{\Gamma_{R,j\epsilon/2} \sim \Gamma_{R,(j-1)\epsilon/2}} \lesssim \int_{\Gamma_{R,\epsilon/2}} |\widehat{\mu}(y)|^2\, dy + \sum_{j=2}^\infty R^{-8(j-1)\epsilon/2} \int_{\Gamma_{R,j\epsilon/2}} |\widehat{\mu}(y)|^2\, dy.$$

Thus (1.5) follows from (2.1) and (2.3).

Turning to the proof of (2.1), we note that by duality (and the fact that μ is finite) it is enough to suppose that f, satisfying $\|f\|_2 = 1$, is supported on $\Gamma_{R,\delta}$ and then to establish the estimate

$$\int |\widehat{f}(y)|^2\, d\mu(y) \lesssim R^{1-\alpha/2+2\delta}\, m^{2-\alpha}. \tag{2.4}$$

The argument we will give differs from the proof of Theorem 3 in [1] only in certain technical details. But because those details are not always obvious, and for the convenience of any reader, we will give the complete proof.

For $y \in \mathbb{R}^2$, write y' for the point on the curve Γ_R which is closest to y (if there are multiple candidates for y', choose the one with least first coordinate). Then $y' = R\gamma(t')$ for some $t' \in [1,2]$. For a dyadic interval $I \subset [1,2]$, define

$$\Gamma_{R,\delta,I} = \{y \in \Gamma_{R,\delta} : t' \in I\},\ f_I = f \cdot \chi_{\Gamma_{R,\delta,I}}.$$

For dyadic intervals $I, J \subset [1,2]$, we write $I \sim J$ if I and J have the same length and are not adjacent but have adjacent parent intervals. The decomposition

$$[1,2] \times [1,2] = \bigcup_{n \geq 2} \Big(\bigcup_{\substack{|I|=|J|=2^{-n} \\ I \sim J}} (I \times J) \Big) \tag{2.5}$$

leads to

$$\int |\widehat{f}(y)|^2 \, d\mu(y) \leq \sum_{n \geq 2} \sum_{\substack{|I|=|J|=2^{-n} \\ I \sim J}} \int |\widehat{f_I}(y)\widehat{f_J}(y)| \, d\mu(y). \tag{2.6}$$

Truncating (2.5) and (2.6) gives

$$\int |\widehat{f}(y)|^2 \, d\mu(y) \leq \tag{2.7}$$
$$\sum_{4 \leq 2^n \leq R^{1/2}} \sum_{\substack{|I|=|J|=2^{-n} \\ I \sim J}} \int |\widehat{f_I}(y)\widehat{f_J}(y)| \, d\mu(y) + \sum_{I \in \mathcal{I}} \int |\widehat{f_I}(y)|^2 \, d\mu(y),$$

where $\mathcal{I}$ is a finitely overlapping set of dyadic intervals I with $|I| \approx R^{-1/2}$.

To estimate the integrals on the right hand side of (2.7), we begin with two geometric observations. The first of these is that if $I \subset [1,2]$ is an interval with length ℓ, then

$$\Gamma_{R,I} \doteq \{R\left(t, \phi(t)\right) : t \in I\}$$

is contained in a rectangle D with side lengths $\lesssim R\ell m, R\ell^2$, which we will abbreviate by saying that D is a $(R\ell m) \times (R\ell^2)$ rectangle. (To see this, note that the since the sine of the angle between vectors $(1, M)$ and $(1, M+\kappa)$ is

$$\frac{\kappa}{\sqrt{1+M^2}\sqrt{1+(M+\kappa)^2}},$$

it follows from (1.4) that the angle between tangent vectors at the beginning and ending points of the curve $\Gamma_{R,I}$ is $\lesssim \ell/m$. Since the distance between these two points is $\lesssim R\ell m$, it is clear that $\Gamma_{R,I}$ is contained in a rectangle D of the stated dimensions.) Secondly, we observe that if $\ell \gtrsim R^{-1/2}$, then an R^δ neighborhood of an $(R\ell m) \times (R\ell^2)$ rectangle is contained in an $(R^{1+\delta}\ell m) \times (R^{1+\delta}\ell^2)$ rectangle. It follows that if I has length $2^{-n} \gtrsim R^{-1/2}$, then the support of f_I is contained in a rectangle D with dimensions $(R^{1+\delta}2^{-n}m) \times (R^{1+\delta}2^{-2n})$.

The next lemma is part of Lemma 3.1 in [1] (the hypothesis $1 \le \alpha \le 2$ there is not necessary for the conclusion of that lemma). To state it, we introduce some notation: ϕ is a nonnegative Schwartz function such that $\phi(x) = 1$ for x in the unit cube Q, $\phi(x) = 0$ if $x \notin 2Q$, and, for each $M > 0$,

$$|\widehat{\phi}| \le C_M \sum_{j=1}^{\infty} 2^{-Mj} \chi_{2^j Q}.$$

For a rectangle $D \subset \mathbb{R}^2$, ϕ_D will stand for $\phi \circ b$, where b is an affine mapping which takes D onto Q.

Lemma 2.1. *Suppose that μ is a non-negative Borel measure on $\mathbb{R}^2$ satisfying* (1.1). *Suppose D is a rectangle with dimensions $R_2 \times R_1$, where $R_2 \gtrsim R_1$, and let D_{dual} be the dual of D centered at the origin. Then, if $\widetilde{\mu}(E) = \mu(-E)$,*

$$(\widetilde{\mu} * |\widehat{\phi_D}|)(y) \lesssim R_2^{2-\alpha},\ y \in \mathbb{R}^2 \tag{2.8}$$

and, if $K \gtrsim 1$, $y_0 \in \mathbb{R}^2$, then

$$\int_{K \cdot D_{dual}} (\widetilde{\mu} * |\widehat{\phi_D}|)(y_0 + y)\, dy \lesssim K^{\alpha} R_2^{1-\alpha} R_1^{-1}. \tag{2.9}$$

Now if $I \in \mathcal{I}$ and $\mathrm{supp} f_I \subset D$ as above, the identity $\widehat{f_I} = \widehat{f_I} * \widehat{\phi_D}$ implies that

$$|\widehat{f_I}| \le (|\widehat{f_I}|^2 * |\widehat{\phi_D}|)^{1/2} \|\widehat{\phi_D}\|_1^{1/2} \lesssim (|\widehat{f_I}|^2 * |\widehat{\phi_D}|)^{1/2}$$

and so

$$\begin{aligned} \int |\widehat{f_I}(y)|^2\, d\mu(y) &\lesssim \int (|\widehat{f_I}|^2 * |\widehat{\phi_D}|)(y)\, d\mu(y) = \\ &\int |\widehat{f_I}(y)|^2 (\widetilde{\mu} * |\widehat{\phi_D}|)(-y)\, dy \lesssim \|f_I\|_2^2\, R^{1-\alpha/2+2\delta} m^{2-\alpha}, \end{aligned} \tag{2.10}$$

where the last inequality follows from (2.8) and the fact that D has dimensions $(R^{1/2+\delta}m) \times R^{\delta}$ since $2^{-n} \approx R^{-1/2}$. Thus the estimate

$$\sum_{I \in \mathcal{I}} \int |\widehat{f_I}(y)|^2\, d\mu(y) \lesssim R^{1-\alpha/2+2\delta} m^{2-\alpha} \sum_{I \in \mathcal{I}} \|f_I\|_2^2 \lesssim R^{1-\alpha/2+2\delta} m^{2-\alpha} \tag{2.11}$$

follows from $\|f\|_2 = 1$ and the finite overlap of the intervals $I \in \mathcal{I}$ (which implies finite overlap for the supports of the $f_I, I \in \mathcal{I}$).

To bound the principal term of the right hand side of (2.7), fix n with $4 \le 2^n \le R^{1/2}$ and a pair I, J of dyadic intervals with $|I| = |J| = 2^{-n}$ and $I \sim J$. Since $I \sim J$, the support of $f_I * f_J$ is contained in a rectangle D with dimensions $(R^{1+\delta}2^{-n}m) \times (R^{1+\delta}2^{-2n})$. For later reference, let v be a unit vector in the direction of the longer side of D. As in (2.10),

$$\int |\widehat{f_I}(y)\widehat{f_J}(y)|\,d\mu(y) \lesssim \int (|\widehat{f_I}\,\widehat{f_J}| * |\widehat{\phi_D}|)(y)\,d\mu(y) = \int |\widehat{f_I}(y)\widehat{f_J}(y)|\,(\widetilde{\mu} * |\widehat{\phi_D}|)(-y)\,dy. \tag{2.12}$$

Now tile $\mathbb{R}^2$ with rectangles P having exact dimensions $C \times (C2^{-n}m^{-1})$ for some large $C > 0$ to be chosen later and having shorter axis in the direction of v. Let ψ be a fixed nonnegative Schwartz function satisfying $\psi(y) = 1$ if $y \in Q$, $\widehat{\psi}(x) = 0$ if $x \notin Q$, and

$$\psi \leq C_M \sum_{j=1}^{\infty} 2^{-Mj} \chi_{2^j Q}. \tag{2.13}$$

Since $\sum_P \psi_P^3 \approx 1$, it follows from (2.12) that if $f_{I,P}$ is defined by

$$\widehat{f_{I,P}} = \psi_P \cdot \widehat{f_I}$$

then

$$\int |\widehat{f_I}(y)\widehat{f_J}(y)|\,d\mu(y) \lesssim \sum_P \Big(\int |\widehat{f_{I,P}}(y)\widehat{f_{J,P}}(y)|^2\,dy\Big)^{1/2} \Big(\int \big|(\widetilde{\mu} * |\widehat{\phi_D}|)(-y)\psi_P(y)\big|^2\,dy\Big)^{1/2}. \tag{2.14}$$

To estimate the first integral in this sum, we begin by noting that the support of $f_{I,P}$ is contained in $\mathrm{supp}(f_I) + P_{\mathrm{dual}}$, where P_{dual} is a rectangle dual to P and centered at the origin. Let $\widetilde{I}$ be the interval with the same center as I but lengthened by $2^{-n}/10$ and let $\widetilde{J}$ be defined similarly. Since $I \sim J$, it follows that $\mathrm{dist}(\widetilde{I}, \widetilde{J}) \geq 2^{-n}/2$. Now the support of f_I is contained in $\Gamma_{R,I} + B(0, R^\delta)$ and P_{dual} has dimensions $(m2^n C^{-1}) \times C^{-1}$ with the longer direction at an angle $\lesssim 2^{-n}/m$ to any of the tangents to the curve $\big(t, \phi(t)\big)$ for $t \in \widetilde{I}$ (or $t \in \widetilde{J}$). Recalling that $2^n \lesssim R^{1/2}$, one can check that, if C is large enough,

$$\mathrm{supp}(f_{I,P}) \subset \Gamma_{R,\widetilde{I}} + B(0, CR^\delta)$$

and, similarly,

$$\mathrm{supp}(f_{J,P}) \subset \Gamma_{R,\widetilde{J}} + B(0, CR^\delta).$$

The following lemma will be proved at the end of this section:

Lemma 2.2. *Suppose ϕ satisfies the estimates*

$$0 < \phi' \leq m_1 \text{ and } \phi'' \geq m_2$$

with $m_1 \geq 1$ and

$$m_1, m_2 \approx m. \tag{2.15}$$

Suppose that the closed intervals $\tilde{I}, \tilde{J} \subset [1,2]$ satisfy dist$(\tilde{I}, \tilde{J}) \geq c\,2^{-n}$. Then, for $\delta > 0$ and $x \in \mathbb{R}^2$, there is the following estimate for the two-dimensional Lebesgue measure of the intersection of translates of tubular neighborhoods of $\Gamma_{R,\tilde{I}}$ and $\Gamma_{R,\tilde{J}}$:

$$\big|\, x + \Gamma_{R,\tilde{I}} + B(0, CR^{\delta}) \ \cap \ \Gamma_{R,\tilde{J}} + B(0, CR^{\delta}) \,\big| \lesssim R^{2\delta} 2^n m. \tag{2.16}$$

The implicit constant in (2.16) *depends only on the implicit constants in* (2.15) *and the positive constants c and C.*

It follows from Lemma 2.2 that for $x \in \mathbb{R}^2$ we have

$$\big|\, x + \mathrm{supp}(f_{I,P}) \ \cap \ \mathrm{supp}(f_{J,P}) \,\big| \lesssim R^{2\delta} 2^n m. \tag{2.17}$$

Now

$$\int |\widehat{f_{I,P}}(y)\widehat{f_{J,P}}(y)|^2 \, dy = \int |\widetilde{f_{I,P}} * f_{J,P}(x)|^2 \, dx$$

and

$$|\widetilde{f_{I,P}} * f_{J,P}(x)| \leq \int |f_{I,P}(w-x)\, f_{J,P}(w)| \, dw \leq$$
$$|x + \mathrm{supp}(f_{I,P}) \cap \mathrm{supp}(f_{J,P})|^{1/2} \left(|\widetilde{f_{I,P}}|^2 * |f_{J,P}|^2(x) \right)^{1/2}.$$

Thus, by (2.17),

$$\Big(\int |\widehat{f_{I,P}}(y)\widehat{f_{J,P}}(y)|^2 \, dy \Big)^{1/2} \lesssim R^{\delta} 2^{n/2} m^{1/2} \Big(\int |\widetilde{f_{I,P}}|^2 * |f_{J,P}|^2(x) \, dx \Big)^{1/2} =$$
$$R^{\delta} 2^{n/2} m^{1/2} \|f_{I,P}\|_2 \|f_{J,P}\|_2. \tag{2.18}$$

To estimate the second integral in the sum (2.14) we use (2.13) to observe that

$$\psi_P \lesssim \sum_{j=1}^{\infty} 2^{-Mj} \chi_{2^j P}.$$

Thus

$$\int (\widetilde{\mu} * |\widehat{\phi_D}|)(-y) \psi_P(y) \, dy \lesssim \sum_{j=1}^{\infty} 2^{-Mj} \int_{2^j P} (\widetilde{\mu} * |\widehat{\phi_D}|)(-y) \, dy.$$

Noting that $2^j P \subset y_P + K D_{\mathrm{dual}}$ for some $K \lesssim R^{1+\delta} 2^{-2n+j}$ and some $y_P \in \mathbb{R}^2$, we apply (2.9) to obtain

$$\int (\widetilde{\mu} * |\widehat{\phi_D}|)(-y) \psi_P(y) \, dy \lesssim$$
$$\sum_{j=1}^{\infty} 2^{-Mj} (R^{1+\delta} 2^{-2n+j})^{\alpha} (R^{1+\delta} 2^{-n} m)^{1-\alpha} (R^{1+\delta} 2^{-2n})^{-1} \lesssim 2^{-n(\alpha-1)} m^{1-\alpha}.$$

Since

$$(\widetilde{\mu} * |\widehat{\phi_D}|)(-y) \lesssim (R^{1+\delta} 2^{-n} m)^{2-\alpha}$$

by (2.8) and since $\psi_P(y) \lesssim 1$, it follows that

$$\Big(\int \big((\widetilde{\mu} * |\widehat{\phi_D}|)(-y)\psi_P(y)\big)^2\,dy\Big)^{1/2} \lesssim R^{1-\alpha/2+\delta(1-\alpha/2)}2^{-n/2}m^{3/2-\alpha}. \tag{2.19}$$

Now (2.18) and (2.19) imply, by (2.14), that

$$\int |\widehat{f}_I(y)\widehat{f}_J(y)|\,d\mu(y) \lesssim R^{1-\alpha/2+\delta(2-\alpha/2)}m^{2-\alpha}\Big(\sum_P \|f_{I,P}\|_2^2\Big)^{1/2}\Big(\sum_P \|f_{J,P}\|_2^2\Big)^{1/2}.$$

Since

$$\sum_P \|\widehat{f_{I,P}}\|_2^2 = \int |\widehat{f}_I(y)|^2 \sum_P |\psi_P(y)|^2\,dy,$$

it follows from $\sum_P \psi_P^2 \lesssim 1$ that

$$\int |\widehat{f}_I(y)\widehat{f}_J(y)|\,d\mu(y) \lesssim R^{1-\alpha/2+\delta(2-\alpha/2)}m^{2-\alpha}\|f_I\|_2\|f_J\|_2.$$

Thus

$$\sum_{\substack{|I|=|J|=2^{-n}\\ I\sim J}} \int |\widehat{f}_I(y)\widehat{f}_J(y)|\,d\mu(y) \lesssim R^{1-\alpha/2+\delta(2-\alpha/2)}m^{2-\alpha}\sum_{\substack{|I|=|J|=2^{-n}\\ I\sim J}} \|f_I\|_2\|f_J\|_2 \lesssim R^{1-\alpha/2+\delta(2-\alpha/2)}m^{2-\alpha}\|f\|_2^2. \tag{2.20}$$

Now (2.4) follows from (2.7), (2.11), (2.20), and the fact that the first sum in (2.7) has $\lesssim \log R$ terms.

Here is the proof of Lemma 2.2:

Proof. Fix $t \in \tilde{I},\ s \in \tilde{J}$ such that

$$x + R\big(t,\phi(t)\big) + \overline{B(0,CR^\delta)}\ \cap\ R\big(s,\phi(s)\big) + \overline{B(0,CR^\delta)} \neq \emptyset \tag{2.21}$$

and such that t is minimal subject to (2.21). Without loss of generality, assume that $t < s$. Suppose that v and w satisfy

$$x + R\big(t+w,\phi(t+w)\big) + \overline{B(0,CR^\delta)}\ \cap\ R\big(s+v,\phi(s+v)\big) + \overline{B(0,CR^\delta)} \neq \emptyset. \tag{2.22}$$

We will begin by observing that

$$w \leq \frac{8C2^nR^{\delta-1}m_1}{c\,m_2}. \tag{2.23}$$

From (2.21) and (2.22) it follows that

$$|w-v|,\ \big|\big(\phi(s+v)-\phi(s)\big)-\big(\phi(t+w)-\phi(t)\big)\big| \leq 4CR^{\delta-1}. \tag{2.24}$$

Now

$$\big(\phi(s+v)-\phi(s)\big)-\big(\phi(t+w)-\phi(t)\big) = \int_t^{t+w}\big(\phi'(u+s-t)-\phi'(u)\big)\,du+e \tag{2.25}$$

where the error term e satisfies $|e| \leq 4CR^{\delta-1}m_1$ because of the first inequality in (2.24) and the bound on ϕ'. Since $s - t \geq c2^{-n}$, the lower bound on ϕ'' shows that the integral in (2.25) exceeds $wc2^{-n}m_2$. Thus if

$$wc2^{-n}\, m_2 > 8CR^{\delta-1}\, m_1$$

(that is, if (2.23) fails) then, since $m_1 \geq 1$, (2.25) exceeds $4CR^{\delta-1}$, contradicting (2.24).

To see (2.16), define $\tilde{t}$ by

$$\tilde{t} = t + \frac{8C2^n R^{\delta-1} m_1}{c\, m_2}$$

and note that by (2.23) the intersection in (2.16) is contained in a translate of

$$\{R\big(u, \phi(u)\big) : t \leq u \leq \tilde{t}\} + B(0, CR^\delta) \doteq \Gamma + B(0, CR^\delta).$$

Using $\phi' \lesssim m$, the length of the curve Γ is $\lesssim 2^n R^\delta m$. Thus Γ is contained in $\lesssim 2^n m$ balls of radius R^δ. This implies (2.16).

□

3. Proof of Theorems 1.2 and 1.3

Proof of Theorem 1.2: First suppose $1 < \alpha < 2$. Choose $\epsilon > 0$ such that $\gamma + 2\epsilon < \alpha(p - 1/2) - p$. Then apply Theorem 1.1 with $\phi(t) = R^{p-1}t^p$ and $m = R^{p-1}$ to conclude that

$$\int_1^2 \left|\widehat{\mu}\big(Rt, (Rt)^p\big)\right|^2 dt \lesssim R^{-\alpha/2+\epsilon} R^{(p-1)(1-\alpha)}$$

and so

$$\int_R^{2R} |\widehat{\mu}(t, t^p)|^2\, t^\gamma\, dt \lesssim R^{-\epsilon}.$$

Now (1.6) follows by taking $R = 2^n$.

To deal with the remaining cases we note that if $d\nu$ is dt on the curve $(t, R^{p-1}t^p)$, $1 \leq t \leq 2$, then there is the estimate $|\widehat{\nu}(\xi)| \lesssim |\xi|^{-1/2}$. It follows from Theorem 1 in [1] that

$$\int_1^2 \left|\widehat{\mu}\big(Rt, (Rt)^p\big)\right|^2 dt \lesssim R^{-\min(\alpha, 1/2)}.$$

This implies the conclusions of Theorem 1.3 in cases (ii) and (iii) exactly as in the preceding paragraph.

Proof of Theorem 1.3: We begin by observing that if the conclusion (1.6) of Theorem 1.2 holds for $\alpha \in (0, 2)$ with C depending only on the size of the support of the nonnegative measure μ and the implied constant in (1.1), then the same conclusion holds (with C replaced by $16\,C$) for complex measures whose total variation measure $|\mu|$ satisfies (1.1).

We consider first the case $\alpha \in (1, 2)$. Suppose R is large and positive. It is easy to check that the set

$$\big\{(t, t^p) : R \leq t \leq R + \sqrt{R}\big\}$$

is contained in a rectangle D with (approximate) dimensions $1 \times R^{p-1/2}$. Let v be a unit vector in the direction of the long axis of D and c_D be the center of D. Also, denote the dual of D centered at the origin by D_{dual}. Note that D_{dual} is a rectangle with dimensions $1 \times R^{1/2-p}$ with short axis in the direction v. Fix a function $\psi \in C_c^\infty$ supported in D_{dual} such that $\widehat{\psi} \gtrsim R^{(p-1/2)(1-\alpha)}$ on D and $\|\psi\|_\infty \lesssim R^{(p-1/2)(2-\alpha)}$. Let $T \approx R^{(p-1/2)(\alpha-1)}$ be a natural number and define μ by

$$\mu(y) \doteq e^{2\pi i y\cdot c_D} \sum_{k=1}^{T} \psi(y - kT^{-1}v). \tag{3.1}$$

It is easy to check that $|\mu|$ satisfies (1.1) independently of R. Also note that

$$|\widehat{\mu}(x)| \gtrsim R^{(p-1/2)(1-\alpha)}\, \chi_D(x) \,\Big|\sum_{k=1}^{T} e^{-2\pi i \frac{k}{T} v\cdot(x-c_D)}\Big|.$$

Now if

$$\Big|\frac{1}{T} v\cdot(x-c_D)\Big| \le 1/4 \pmod 1,$$

then we have

$$\Big|\sum_{k=1}^{T} e^{-2\pi i \frac{k}{T} v\cdot(x-c_D)}\Big| \gtrsim T.$$

Therefore there are $N \approx R^{p-1/2}/T \approx R^{(p-1/2)(2-\alpha)}$ subrectangles $P_1, ..., P_N$ of D with dimensions $1\times 1/4$ whose centers are in an arithmetic progression with distance T between the adjacent points such that

$$|\widehat{\mu}(x)| \gtrsim R^{(p-1/2)(1-\alpha)}\, T \sum_{k=1}^{N} \chi_{P_k}(x) \approx \sum_{k=1}^{N} \chi_{P_k}(x).$$

Using this we obtain

$$\begin{aligned}\int_R^{R+\sqrt{R}} |\widehat{\mu}(t,t^p)|^2\, t^\gamma\, dt &\gtrsim R^\gamma \int_R^{R+\sqrt{R}} \sum_{k=1}^{N} \chi_{P_k}(t,t^p)\, dt \\ &\gtrsim R^\gamma \frac{N}{R^{p-1}} \approx R^{\gamma-\alpha p+\alpha/2+p}.\end{aligned}$$

This implies that $\gamma \le \alpha p - \alpha/2 - p$ and so gives the conclusion (i) of Theorem 1.3.

The conclusion (ii) of Theorem 1.3 also follows from the examples just constructed: since the support of μ above is contained in a ball of radius ≈ 1, if $|\mu|$ satisfies (1.1) for some $\alpha > 1$, then the same is certainly true for all $\alpha \in (0,1]$. Taking $\alpha = 1+\delta$ for arbitrary $\delta > 0$ gives $\gamma \le -1/2$.

To conclude, suppose $\alpha \in (0,1/2)$ and $R > 0$ is large. Let D be a rectangle with dimensions $R \times R^p$ which contains

$$\big\{(t,t^p) : R \le t \le 2R\big\},$$

and let v, C_D, and $D_{\rm dual}$ be as above. Note that now $D_{\rm dual}$ is a rectangle with dimensions $R^{-1} \times R^{-p}$ with short axis in the direction v. Fix a function $\psi \in C_c^\infty$ supported in $D_{\rm dual}$ and satisfying $\widehat{\psi} \gtrsim R^{-\alpha}$ on D and $\|\psi\|_\infty \lesssim R^{p+1-\alpha}$. Fix a natural number T with $T \approx R^\alpha$ and again define μ by (3.1). As before, $|\mu|$ satisfies (1.1) independently of R and there are $N \approx R^p/T \approx R^{p-\alpha}$ disjoint subrectangles $P_1, ..., P_N$ of D of dimensions $1 \times 1/4$ such that

$$|\widehat{\mu}(x)| \gtrsim R^{-\alpha}\, T \sum_{k=1}^{N} \chi_{P_k}(x) \approx \sum_{k=1}^{N} \chi_{P_k}(x).$$

As above, that leads to

$$\begin{aligned}\int_R^{2R} |\widehat{\mu}(t,t^p)|^2\, t^\gamma\, dt &\gtrsim R^\gamma \int_R^{2R} \sum_{k=1}^{N} \chi_{P_k}(t,t^p)\, dt \\ &\gtrsim R^\gamma \frac{N}{R^{p-1}} \approx R^{\gamma+p-\alpha-(p-1)}.\end{aligned}$$

This gives the conclusion (iii) of Theorem 1.3.

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