ON FALCONER'S DISTANCE SET CONJECTURE

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ABSTRACT. In this paper, using a recent parabolic restriction estimate of Tao, we obtain improved partial results in the direction of Falconer's distance set conjecture in dimensions $d \geq 3$.

1. Introduction

Let E be a compact subset of $\mathbf{R}^d$. The distance set, $\Delta(E)$, of E is defined as

$$\Delta(E) = \{|x - y| : x, y \in E\}.$$

Erdös' famous distinct distances conjecture [7] states that for any $\varepsilon > 0$ and for any finite set $E \subset \mathbf{R}^d$, $d \geq 2$,

$$\#\Delta(E) \geq C_{d,\varepsilon}(\#E)^{\frac{2}{d}-\varepsilon}.$$

This conjecture is still open in all dimensions $d \geq 2$. For various partial results and references see [17], [1] and [13].

Falconer's conjecture [8] is a variant of Erdös' conjecture:

Conjecture. Let $d \geq 2$. Let E be a compact subset of $\mathbf{R}^d$. Then,

$$\dim(E) > \frac{d}{2} \implies |\Delta(E)| > 0.$$

Here $|\cdot|$ is the Lebesgue measure and $\dim(\cdot)$ is the Hausdorff dimension.

Like Erdös' conjecture, Falconer's conjecture is open in every dimension. In [8], Falconer gave an example showing that $\frac{d}{2}$ in the conjecture is optimal and proved that $\dim(E) > \frac{d+1}{2}$ implies $|\Delta(E)| > 0$. Bourgain [3] improved this result in every dimension, and in particular proved that in $\mathbf{R}^2$, $\dim(E) > \frac{13}{9}$

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suffices. Later, Wolff [24] proved that in $\mathbf{R}^2$, $\dim(E) > \frac{4}{3}$ suffices. In [6], the author obtained a simplified proof of Wolff's result and noted that it is possible to obtain the following improved partial result in higher dimensions using the method in [6] and a bilinear Fourier restriction estimate by Tao [22]. In this paper, we prove

Theorem 1. *Let $d \geq 2$. Let E be a compact subset of $\mathbf{R}^d$ with*

$$\dim(E) > \frac{d(d+2)}{2(d+1)}.$$

Then $|\Delta(E)| > 0$.

There are other positive results in the direction of Falconer's conjecture. For example, Mattila [14] proved that in $\mathbf{R}^2$, $\dim(E) > 1$ implies $\dim(\Delta(E)) \geq \frac{1}{2}$. Recently, Bourgain [4] improved this result and proved that there exists $c > 0$ such that in $\mathbf{R}^2$, $\dim(E) > 1$ implies $\dim(\Delta(E)) > \frac{1}{2} + c$. Bourgain's result relies on a paper by Katz and Tao [12] which relates the Falconer's conjecture to various other problems in harmonic analysis.

There are lots of variations of Falconer's problem. Notably, Mattila and Sjölin [16] proved that $\Delta(E)$ has interior points if $\dim(E) > \frac{d+1}{2}$. Peres and Schlag [18] considered pinned distance sets,

$$\Delta(x, E) = \{|x - y| : y \in E\},$$

and proved that if $\dim(E) > \frac{d+1}{2}$ then $|\Delta(x, E)| > 0$ for almost every $x \in E$.

One can also consider distance sets with respect to general metrics. Let K be a convex symmetric body in $\mathbf{R}^d$, $d \geq 2$. Define $\Delta_K(E) = \{d_K(x, y) : x, y \in E\}$, where d_K is the distance induced by K. Iosevich and Laba [10] investigated the relation between the curvature of the boundary of K and the size of the distance sets. Hofmann and Iosevich [9] (also see [2] for a similar result in higher dimensions) proved that in $\mathbf{R}^2$ if $\dim(E) > 1$ then $|\Delta_K(E)| > 0$ for almost every ellipse K centered at the origin. We note that our main result, Theorem 1, remains valid for Δ_K in the case when the boundary of K is smooth and has non-vanishing Gaussian curvature (see Remark 1 below).

List of notations.

χ_A: characteristic function of the set A.

$B(x,r) := \{y : |x-y| < r\}$.

$d(A,B)$: the distance between the sets A and B.

$A_R(C) := \{x \in \mathbf{R}^d : ||x| - R| \leq C\}$.

C: a constant which may vary from line to line.

$A \lesssim B$: $A \leq CB$.

$A \approx B$: $A \lesssim B$ and $B \lesssim A$.

$A \ll B$: $A \leq \frac{1}{C}B$, for some large constant C.

$|A|$: length of the vector A or the measure of the set A.

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2. Mattila's approach to distance set problem

In [14], Mattila developed a method to attack the distance set problem. For a very good exposition of this method, see [26]. Mattila's approach was used in [14, 3, 24, 9, 6, 2].

Let μ be a probability measure supported in E. Let ν_μ be the push forward of $\mu \times \mu$ under the distance map $(x,y) \mapsto |x-y|$, i.e.,

$$\nu_\mu(A) = \mu \times \mu(\{(x,y) : |x-y| \in A\}), \text{ for Borel sets } A \subset \mathbf{R}.$$

It is easy to check that ν_μ is a probability measure supported in $\Delta(E)$. Note that if the Fourier transform of ν_μ,

$$\widehat{\nu_\mu}(\xi) := \int e^{-ix\cdot\xi} \mathrm{d}\nu_\mu(x),$$

is an L^2 function, then ν_μ should be absolutely continuous with an L^2 density and hence

$$|\Delta(E)| \geq |\mathrm{Supp}(\nu_\mu)| > 0.$$

Using this idea and the Fourier asymptotics of the surface measure of the unit sphere in $\mathbf{R}^d$, Mattila proved [14]:

Theorem A. *Let $\alpha \in (0,d)$. Let E be a compact subset of $\mathbf{R}^d$ with $\dim(E) > \alpha$. Assume that there is a probability measure μ supported in E such that*

$$\|\widehat{\mu}(R\cdot)\|_{L^2(S^{d-1})} \leq C_\mu R^{\frac{\alpha-d}{2}}, \quad \forall R > 1. \tag{1}$$

Then $|\Delta(E)| > 0$.

Note that Theorem A proves the distance set conjecture for Salem sets [19, 11]. A set $E \subset \mathbf{R}^d$ is called a Salem set if for each $\beta < \dim(E)$, there exists a probability measure μ supported in E such that

$$|\widehat{\mu}(\xi)| \lesssim |\xi|^{-\frac{\beta}{2}}, \quad \forall \xi \in \mathbf{R}^d.$$

To apply Theorem A to arbitrary compact sets, one needs Frostman's lemma (see, e.g., [15]).

Definition 1. *A compactly supported probability measure μ is called α-dimensional if it satisfies*

$$\mu(B(x,r)) \leq C_\mu r^\alpha, \quad \forall r > 0, \forall x \in \mathbf{R}^d. \tag{2}$$

Frostman's Lemma. *If E is a compact subset of $\mathbf{R}^d$ with $\dim(E) > \alpha$, then there is an α-dimensional measure μ supported in E.*

Frostman's lemma and Mattila's theorem imply:

Lemma 2.1. *Fix $\alpha \in (0,d)$. Assume that the inequality (1) holds for all α-dimensional measures. Then for any compact $E \subset \mathbf{R}^d$*

$$\dim(E) > \alpha \implies |\Delta(E)| > 0.$$

In view of Lemma 2.1, Theorem 1 is a corollary of the following:

Theorem 2. *Let $d \geq 2$ and $\alpha \in (0, d)$. Let μ be an α-dimensional measure. Then for each $q > \frac{d+2}{d}$,*

$$\|\widehat{\mu}(R\cdot)\|_{L^2(S^{d-1})} \leq C_{q,\mu} R^{-\frac{\alpha}{2q}}, \quad \forall R > 1.$$

Like Theorem 1, Theorem 2 was first proved in [24] for $d = 2$. Under the hypothesis of Theorem 2, it is also known that [14, 20] (also see [21, 6])

$$\|\widehat{\mu}(R\cdot)\|_{L^2(S^{d-1})} \lesssim R^{-\max(\frac{\alpha-1}{2}, \min(\frac{\alpha}{2}, \frac{d-1}{4}))}, \quad \forall R > 1. \tag{3}$$

Theorem 2 and (3) give optimal bounds for each $\alpha \in (0, 2)$ for $d = 2$ (see, e.g., [20, 24, 6]). Therefore, one can not improve the result in Theorem 1 for $d = 2$ using Mattila's approach. In higher dimensions, (3) is optimal for $\alpha \leq \frac{d-1}{2}$ (see [20]); however, there is no reason to believe that Theorem 2 and (3) give optimal bounds for $\alpha > \frac{d-1}{2}$.

It is essential that in Theorem 2, we are averaging $\widehat{\mu}(R\cdot)$ on a surface with non-vanishing Gaussian curvature. In general, the Fourier transform $\widehat{\mu}(\xi)$ of an α-dimensional measure μ does not have to converge to zero as $|\xi| \to \infty$. In fact, for any $d \geq 1$ and for any $\alpha \in (0, d)$, there are Cantor-type measures in $\mathbf{R}^d$ of dimension greater than α whose Fourier transform does not converge to 0 at infinity [19].

Remark 1. Mattila's approach can be modified for distance sets with respect to general metrics. Let K be a convex symmetric body. Assume that the boundary of K is smooth and has non-vanishing Gaussian curvature. Let K^* be the dual of K. One can modify Mattila's approach and prove that the statement of Lemma 2.1 remains valid if $\Delta(E)$ is replaced with $\Delta_K(E)$ and S^{d-1} in (1) is replaced with ∂K^* (see [9, 2]). We note that Theorem 2 remains valid, too, if we replace S^{d-1} with ∂K^*. The proof of this fact follows the same line below with minor changes in the statements and proofs of Corollary 2 and Lemma 5.2. Therefore, Theorem 1 holds for Δ_K if K has a smooth boundary with non-vanishing Gaussian curvature.

3. Tao's bilinear parabolic extension estimate

In the proof of Theorem 2, we use a bilinear restriction estimate for elliptic surfaces by Tao [22]. First let us recall the definition of elliptic surfaces from [23]:

Definition 2. *We say* $\phi : B(0,1) \subset \mathbf{R}^{d-1} \to \mathbf{R}$ *is an* (M, ε_0)*-elliptic phase if* ϕ *satisfies*
i) $\|\phi\|_{C^\infty} < M$,
ii) $\phi(0) = \nabla\phi(0) = 0$, *and*
iii) For all $x \in B(0,1)$, *all eigenvalues of the Hessian* $\phi_{x_i x_j}(x)$ *lie in* $[1 - \varepsilon_0, 1 + \varepsilon_0]$.
We say S *is an* (M, ε_0)*-elliptic surface if* $S = \{(x, y) \in B(0,1) \times \mathbf{R} \subset \mathbf{R}^d : y = \phi(x)\}$ *for some* (M, ε_0)*-elliptic phase* ϕ.

Note that in this definition the term "elliptic" is used in a slightly non-standard way. In classical PDE, a non-vanishing symbol is considered to be elliptic. In the definition above, the non-vanishing of the curvature is required, too, (see II below). A model example for an elliptic phase is $\phi(x) = \frac{|x|^2}{2}$. We recall the following properties of elliptic phases (see, e.g., [23]):
I) Let ϕ be an (M, ε_0)-elliptic phase and $B(x_0, \eta) \subset B(0,1)$. Let

$$\tilde{\phi}(x) := \frac{1}{\eta^2} \left(\phi(x\eta + x_0) - \phi(x_0) - \eta x \cdot \nabla\phi(x_0) \right), \quad x \in B(0,1).$$

Then $\tilde{\phi}$ is a $(C_d M, \varepsilon_0)$-elliptic phase.
II) Let S be a smooth compact submanifold of $\mathbf{R}^d$ with strictly positive principal curvatures. Note that for any $\varepsilon_0 > 0$ and for any $s \in S$ there is a neighborhood U_s of s and an affine bijection a_s of $\mathbf{R}^d$ such that $a_s(U_s)$ is an (M, ε_0)-elliptic surface, where M depends only on d, $\|\phi\|_{C^\infty}$ and the principal curvatures at s. Moreover, by using a partition of unity, we can write S as a union of affine images of finitely many (M, ε_0)-elliptic surfaces.

These observations are especially important for the extension of Theorem 2 to ∂K^* (see Remark 1 above).

The following theorem is proved in [22] for $d \geq 3$. The $d = 2$ case is basically the Carleson-Sjölin Theorem [5]. In [6], it was used in the proof of Theorem 2 for $d = 2$.

Theorem B. *Let $d \geq 2$. For any $M > 0$, there exists $\varepsilon_0 > 0$ such that the following statement holds.*
Let S_1, S_2 be compact subsets of an (M, ε_0)-elliptic surface in $\mathbf{R}^d$ with $d(S_1, S_2) > \frac{1}{2}$. Let σ_j be the Lebesgue measure on S_j, $j = 1, 2$. Then for all $q > \frac{d+2}{d}$, we have

$$\|\widehat{f_1 d\sigma_1}\widehat{f_2 d\sigma_2}\|_{L^q(\mathbf{R}^d)} \leq C_{M,q,d}\|f_1\|_{L^2(S_1,d\sigma_1)}\|f_2\|_{L^2(S_2,d\sigma_2)}, \tag{4}$$

for all $f_j \in L^2(S_j, d\sigma_j)$, $j = 1, 2$.

In [22], this theorem is proved explicitly only for the paraboloid. The version we stated here can be proved similarly, see the last section of [22] where the necessary modifications are described.

We need the following scaled and mollified version of this theorem (see, e.g., [23]). In view of II) above, choose N_d large enough so that any subset of S^{d-1} of diameter $\lesssim \frac{1}{N_d}$ is an affine image of an elliptic surface which satisfies the hypothesis of Theorem B. Let $A_R(\varepsilon)$ denote the set $\{x \in \mathbf{R}^d : ||x| - R| \leq \varepsilon\}$.

Corollary 1. *Fix a spherical cap U in $A_1(\varepsilon)$, $(\varepsilon \ll 1/N_d)$, of diameter $\lesssim 1/N_d$. If I_1 and I_2 are subsets of U of diameter η with $d(I_1, I_2) \approx \eta$, then for $q > \frac{d+2}{d}$, we have*

$$\|\widehat{f_1}\widehat{f_2}\|_{L^q(\mathbf{R}^d)} \leq C_{q,d}\varepsilon\eta^{d-1-\frac{d+1}{q}}\|f_1\|_2\|f_2\|_2,$$

for all $f_j \in L^2(I_j)$, $j = 1, 2$.

Proof. First note that the inequality (4) is invariant under translations of one or both of the surfaces S_1, S_2. Therefore, under the hypothesis of

Theorem B, we have

$$\|\widehat{f_1}\widehat{f_2}\|_{L^q(\mathbf{R}^d)} \lesssim \varepsilon \|f_1\|_2 \|f_2\|_2, \tag{5}$$

for all $f_j \in L^2(S_j^\varepsilon)$, $j = 1, 2$, where S_j^ε is the ε-neighborhood of S_j. This follows easily from the definition of Lebesgue measure.

Let e be the unit vector in the direction of the center of mass of $I_1 \cup I_2$. Let $\{e_1 = e, e_2, ..., e_d\}$ be an orthogonal basis for $\mathbf{R}^d$. Let $T : \mathbf{R}^d \to \mathbf{R}^d$ be the linear map which satisfies

$$T(e_1) = \frac{1}{\eta^2} e_1, \quad T(e_j) = \frac{1}{\eta} e_j, \;\; j = 2, 3, ..., d,$$

In view of I) and II) above, $C_j = TI_j$ is contained in $\approx \frac{\varepsilon}{\eta^2}$-neighborhood of an affine image of a surface S_j, $j = 1, 2$, where the surfaces S_1, S_2 satisfy the hypothesis of Theorem B (with M independent of η, I_1, I_2).

Let $g_j(x) = f_j(T^{-1}x)$, $j = 1, 2$. Since g_j is supported in C_j, using (5) we obtain

$$\|\widehat{g_1}\widehat{g_2}\|_q \lesssim \frac{\varepsilon}{\eta^2} \|g_1\|_2 \|g_2\|_2. \tag{6}$$

The following elementary identities and (6) yield the claim of the corollary:

$$\widehat{f_j}(\xi) = \frac{1}{\det(T)} \widehat{g_j}(T^{-1}(\xi)) = \eta^{d+1} \widehat{g_j}(T^{-1}(\xi)), \;\; j = 1, 2,$$
$$\|\widehat{f_1}\widehat{f_2}\|_q = \eta^{(d+1)(2-\frac{1}{q})} \|\widehat{g_1}\widehat{g_2}\|_q,$$
$$\|f_j\|_2 = \eta^{\frac{d+1}{2}} \|g_j\|_2, \;\; j = 1, 2.$$

□

The following Corollary is obtained from Corollary 1 using a dilation:

Corollary 2. *If I_1 and I_2 are subsets of $A_R(\varepsilon)$, ($\varepsilon \ll R/N_d$), of diameter $\eta \lesssim R/N_d$ with $d(I_1, I_2) \approx \eta$, then for $q > \frac{d+2}{d}$, we have*

$$\|\widehat{f_1}\widehat{f_2}\|_{L^q(\mathbf{R}^d)} \leq C_{q,d} \varepsilon R^{\frac{1}{q}} \eta^{d-1-\frac{d+1}{q}} \|f_1\|_2 \|f_2\|_2, \tag{7}$$

for all $f_j \in L^2(I_j)$, $j = 1, 2$.

4. Uncertainty principle

Let φ be a Schwartz function satisfying

$$\varphi(\xi) = 1, \text{ for } |\xi| < 2 \text{ and } \varphi(\xi) = 0, \text{ for } |\xi| > 4.$$

Let D be a ball of radius s in $\mathbf{R}^d$. Fix an affine bijection a_D of $\mathbf{R}^d$ which maps D to $B(0,1)$. Let $\varphi_D = \varphi \circ a_D$. Since φ is a Schwartz function, for each $M \in \mathbf{N}$, we have

$$|\varphi_D^\vee(x)| = s^d |\varphi^\vee(sx)| \le C_{M,d} s^d \sum_{j=1}^{\infty} 2^{-Mj} \chi_{B(0,2^j s^{-1})}(x), \quad \forall x \in \mathbf{R}^d. \tag{8}$$

The following well-known corollary of the uncertainty principle (see, e.g., [26, Chapter 5]) is another important ingredient of the proof of Theorem 2. We give a proof for the sake of completeness.

Lemma 4.1. *Let μ be an α-dimensional measure in $\mathbf{R}^d$. Let D be a ball of radius s in $\mathbf{R}^d$. Then the function $\mu_D := |\varphi_D^\vee| * \mu$ satisfies*
i) $\|\mu_D\|_\infty \lesssim s^{d-\alpha}$,
ii) $\|\mu_D\|_1 \lesssim 1$,
iii) $\mu_D(\mathcal{B}) := \int_{\mathcal{B}} \mu_D(y)dy \lesssim r^\alpha$, for any ball $\mathcal{B}$ of radius $r \ge 100s^{-1}$.

Proof. i) Fix $M > 100d$. Using (8) and (2), we obtain

$$\begin{aligned} 0 \le \mu_D(x) &\lesssim s^d \sum_{j=1}^{\infty} 2^{-Mj} \int \chi_{B(0,2^j s^{-1})}(x-y) \mathrm{d}\mu(y) \\ &\lesssim s^d \sum_{j=1}^{\infty} 2^{-Mj} (2^j s^{-1})^\alpha \lesssim s^{d-\alpha}. \end{aligned}$$

ii) follows from Young's inequality and the observation $\|\varphi_D^\vee\|_1 \lesssim 1$.
iii) Using (8), we get

$$\mu_D(\mathcal{B}) \lesssim s^d \sum_{j=1}^{\infty} 2^{-Mj} \int\int \chi_{\mathcal{B}}(y) \chi_{B(0,2^j s^{-1})}(y-u) \mathrm{d}\mu(u)\mathrm{d}y$$

Note that $y \in \mathcal{B}$ and $y - u \in B(0, 2^j s^{-1})$ imply $u \in \mathcal{B} + B(0, 2^j s^{-1})$. Using this, Fubini's theorem and then (2), we obtain

$$\begin{aligned}
\mu_D(\mathcal{B}) &\lesssim s^d \sum_{j=1}^{\infty} 2^{-Mj} \int\int \chi_{\mathcal{B}+B(0,2^j s^{-1})}(u) \chi_{B(0,2^j s^{-1})}(y-u) \mathrm{d}y \mathrm{d}\mu(u) \\
&\lesssim s^d \sum_{j=1}^{\infty} 2^{-Mj} (r + 2^j s^{-1})^{\alpha} (2^j s^{-1})^d \\
&\lesssim \sum_{j=1}^{\infty} 2^{-\frac{Mj}{2}} (r + 2^j s^{-1})^{\alpha} \lesssim r^{\alpha}.
\end{aligned}$$

□

5. Proof of Theorem 2

The proof is similar to the proof given in [6]. As in [24, 6], we work with the dual formulation:

Lemma 5.1. *Theorem 2 follows from the following statement: For all $q > \frac{d+2}{d}$, for all α-dimensional measures μ, for all $R > 1$ and for all f supported in $A_R(1)$, we have*

$$\Big| \int f^{\vee}(u) d\mu(u) \Big| \leq C_{q,\mu} R^{\frac{d-1}{2} - \frac{\alpha}{2q}} \|f\|_2, \tag{9}$$

where $f^{\vee}$ is the inverse Fourier transform of f.

Proof. [24] Fix $q_0 > \frac{d+2}{d}$. Note that by duality, Fubini's theorem and the statement of the lemma, we have

$$\begin{aligned}
\|\widehat{\mu}\|_{L^2(A_R(1))} &= \sup_{\|f\|_{L^2(A_R(1))}=1} \Big| \int_{A_R(1)} f(u) \widehat{\mu}(u) du \Big| \\
&= \sup_{\|f\|_{L^2(A_R(1))}=1} \Big| \int \widehat{f}(u) d\mu(u) \Big| \\
&\leq C_{q,\mu} R^{\frac{d-1}{2} - \frac{\alpha}{2q}}, \quad \forall R > 1.
\end{aligned}$$

This easily implies that for any $0 < \varepsilon \ll 1$,

$$\|\widehat{\mu}\|_{L^2(A_R(R^{\varepsilon}))} \leq C_{q,\mu} R^{\frac{d-1}{2} - \frac{\alpha}{2q} + C\varepsilon}, \quad \forall R > 1. \tag{10}$$

Take a Schwartz function ϕ equal to 1 in the support of μ. Note that $\widehat{\mu} = \widehat{\mu} * \widehat{\phi}$. Let $d\sigma_R$ be the surface measure on RS^{d-1}. We have

$$\begin{aligned} \|\widehat{\mu}(R\cdot)\|^2_{L^2(S^{d-1})} &= C_d R^{-(d-1)} \|\widehat{\mu}\|^2_{L^2(RS^{d-1})} = C_d R^{-(d-1)} \left\| \widehat{\mu} * \widehat{\phi} \right\|^2_{L^2(RS^{d-1})} \\ &\le C_d R^{-(d-1)} \left\| \widehat{\phi} \right\|_1 \left\| |\widehat{\mu}|^2 * \widehat{\phi} \right\|_{L^1(RS^{d-1})} \\ &\lesssim R^{-(d-1)} \int |\widehat{\mu}|^2(u)(|\widehat{\phi}| * d\sigma_R)(u) du \\ &\lesssim R^{-(d-1)} \int |\widehat{\mu}|^2(u)(1+|R-|u||)^{-M} du. \end{aligned} \tag{11}$$

The second line follows from Cauchy-Schwarz inequality (as in (15) below); the third line from Fubini's theorem and the last line from the Schwartz decay of ϕ. Here M is a large constant and the implicit constants in the inequalities depend on d, μ, ϕ, and M. Choose $q \in ((d+2)/2, q_0)$. Using (10) for small $\varepsilon = \varepsilon(d, \alpha, q, q_0)$ and (11) for large $M = M(\varepsilon, d, q, q_0, \alpha)$, we obtain

$$\begin{aligned} \|\widehat{\mu}(R\cdot)\|^2_{L^2(S^{d-1})} &\lesssim R^{-(d-1)} \left[\|\widehat{\mu}\|^2_{L^2(A_R(R^\varepsilon))} + \int_{A_R(R^\varepsilon)^c} (1+|R-|u||)^{-M} du \right] \\ &\lesssim R^{-\frac{\alpha}{q}+2C\varepsilon} + R^{-M\varepsilon/2} \lesssim R^{-\frac{\alpha}{q_0}}. \end{aligned}$$

This yields Theorem 2 and hence finishes the proof of the lemma. □

Let f be as in Lemma 5.1 with L^2 norm 1. Below, we prove that

$$\|f^\vee\|_{L^2(\mathrm{d}\mu)} \lesssim R^{\frac{d-1}{2} - \frac{\alpha}{2q}}. \tag{12}$$

(9) can be obtained from (12) using Cauchy-Schwarz inequality. As in [6], we use the bilinear approach. It suffices to prove (12) for functions f supported in a subset of $A_R(1)$ of diameter $\ll R$. Consider a dyadic decomposition of $A_R(1)$ into spherical caps, I, with dimensions $2 \times 2^n \times \ldots \times 2^n$ for

$$R^{\frac{1}{2}} \ll 2^n \ll R.$$

We say I has sidelength 2^n and write $\ell(I) = 2^n$. The unique cap of sidelength 2^{n+1} which contains I is called the parent of I. Let I and J be caps with the same sidelength. We say I and J are related, $I \sim J$, if they are not adjacent but their parents are.

Let $f_I := f\chi_I$. As in [6], we have

$$\|f^\vee\|^2_{L^2(\mathrm{d}\mu)} \leq \sum_{R^{\frac12} \ll 2^n \ll R} \sum_{\ell(I)=2^n, I\sim J} \|f_I^\vee f_J^\vee\|_{L^1(\mathrm{d}\mu)} + \sum_{I\in I_E} \|f_I^\vee\|^2_{L^2(\mathrm{d}\mu)} =: S_1 + S_2. \tag{13}$$

Here I_E is a set of dyadic caps with sidelengths $\approx R^{\frac12}$ satisfying the finite overlapping property:

$$\Big\| \sum_{I\in I_E} \chi_I \Big\|_\infty \lesssim 1. \tag{14}$$

First, we obtain a bound for S_2. Since each $I \in I_E$ is contained in a ball D of radius $CR^{\frac12}$, we have $f_I^\vee = f_I^\vee * \varphi_D^\vee$, ($\varphi_D$ is defined in the beginning of Section 4). Using this and Cauchy-Schwarz inequality, we have

$$|f_I^\vee| \leq (|f_I^\vee|^2 * |\varphi_D^\vee|)^{\frac12} \|\varphi_D^\vee\|_1^{\frac12} \lesssim (|f_I^\vee|^2 * |\varphi_D^\vee|)^{\frac12}. \tag{15}$$

Using this, Fubini's theorem and Lemma 4.1, we obtain

$$\|f_I^\vee\|^2_{L^2(\mathrm{d}\mu)} \leq \int |f_I^\vee(x)|^2 (\mu * |\varphi_D^\vee|)(x) dx \lesssim \|f_I^\vee\|_2^2 R^{\frac{d-\alpha}{2}} = \|f_I \vee\|_2^2 R^{\frac{d-\alpha}{2}}. \tag{16}$$

Using (16) and (14), we obtain

$$S_2 = \sum_{I\in I_E} \|f_I^\vee\|^2_{L^2(\mathrm{d}\mu)} \lesssim R^{\frac{d-\alpha}{2}} \sum_{I\in I_E} \|f_I\|_2^2 \lesssim R^{\frac{d-\alpha}{2}} \|f\|_2^2 = R^{\frac{d-\alpha}{2}}.$$

This term is harmless since $\frac{d-\alpha}{2} < d - 1 - \frac{\alpha}{q}$, for $\alpha \in (0, d)$ and $q > \frac{d+2}{d}$.

In the remaining part of the paper we prove that for $q > \frac{d+2}{d}$, $S_1 \lesssim R^{d-1-\frac{\alpha}{q}}$. Fix n and $I \sim J$ with $|I| = |J| = 2^n$. First, we prove that

$$\|f_I^\vee f_J^\vee\|_{L^1(\mathrm{d}\mu)} \leq C_{\alpha,q,d} R^{d-1-\frac{\alpha}{q}} \|f_I\|_2 \|f_J\|_2. \tag{17}$$

Note that $I+J$ is contained in a ball of radius $C2^n$. Hence, $f_I * f_J$ is supported in a ball D of radius $C2^n$. Using this as in (16), we obtain

$$\|f_I^\vee f_J^\vee\|_{L^1(\mathrm{d}\mu)} \leq \int |f_I^\vee(x) f_J^\vee(x)| \mu_D(x) dx, \tag{18}$$

where $\mu_D = \mu * |\varphi_D^\vee|$.

Let e be the unit vector which is in the direction of the center of mass of $I \cup J$. Consider a tiling of $\mathbf{R}^d$ with rectangles P of dimensions $100 \times 100 \frac{2^n}{R} \times \ldots \times 100 \frac{2^n}{R}$, the long axis being in the direction e. For each P, let a_P be an affine bijection from $\mathbf{R}^d$ to $\mathbf{R}^d$ which maps P to the unit cube. Let ϕ be a Schwartz function satisfying

$$\phi(x) \geq \chi_{B(0,1)}(x),\ x \in \mathbf{R},\ \text{ and } \quad \mathrm{supp}(\widehat{\phi}) \subset B(0,1). \tag{19}$$

Let $\phi_P := \phi \circ a_P$ and $f_{I,P} := \widehat{f_I^\vee \phi_P}$. Using (19) and the fact that the rectangles P tile $\mathbf{R}^d$, we obtain

$$\begin{aligned}(18) &\lesssim \sum_P \int |f_{I,P}^\vee(x) f_{J,P}^\vee(x)| \mu_D(x) \phi_P(x) \mathrm{d}x \\ &\lesssim \sum_P \|f_{I,P}^\vee f_{J,P}^\vee\|_q \|\mu_D \phi_P\|_{q'}, \end{aligned} \tag{20}$$

where $q > \frac{d+2}{d}$ and $q' = \frac{q}{q-1}$.

To estimate $\|f_{I,P}^\vee f_{J,P}^\vee\|_q$, we use the Corollary 2 of Tao's theorem. Let I_P be the support of $f_{I,P}$. Note that I_P is contained in $I + \mathrm{supp}(\widehat{\phi_P}) \subset I + P_{dual}$, where P_{dual} is the dual of P centered at the origin. We have

Lemma 5.2. *$I + P_{dual}$ is contained in a spherical cap of dimensions $10 \times \frac{11}{10} 2^n \times \ldots \times \frac{11}{10} 2^n$ in $A_R(10)$ which contains I.*

Proof. Note that P_{dual} is a rectangle of dimensions $100^{-1} \times 100^{-1} R 2^{-n} \times \ldots \times 100^{-1} R 2^{-n}$, the short axis being in the direction e. For each $p \in P_{dual}$ and $x \in I$, the angle between $p - e\langle p, e\rangle$ and the hyperplane H_x with normal x is $\leq 10 \frac{2^n}{R}$. Therefore P_{dual} is contained in $\frac{1}{10}$-neighborhood of $H_x \cap B(0, 100^{-1} R 2^{-n})$. Note that if $|x| \approx R$, and $r \ll R^{\frac{1}{2}}$, then $x +$

$(H_x \cap B(0,r))$ is contained in a spherical cap containing x of dimensions $\approx 1 \times r \times ... \times r$ in $A_{|x|}(1)$. This finishes the proof since

$$100^{-1}R2^{-n} \leq 100^{-1}R^{\frac{1}{2}} \ll 2^n.$$

□

Using Lemma 5.2 for I and J, we see that I_P and J_P have diameter $\lesssim 2^n$; they are contained in $A_R(10)$ and $d(I_P, J_P) \gtrsim 2^n$. Therefore, Corollary 2 implies that

$$\|f_{I,P}^\vee f_{J,P}^\vee\|_q \lesssim R^{\frac{1}{q}} 2^{n(d-1-\frac{d+1}{q})} \|f_{I,P}\|_2 \, \|f_{J,P}\|_2. \tag{21}$$

We bound $\|\mu_D\phi_P\|_{q'}$ by interpolating between L^1 and L^∞. Using the Schwarz decay of ϕ_P, we have

$$\|\mu_D\phi_P\|_1 \leq \sum_{j=1}^\infty 2^{-Mj} \int \mu_D(x)\chi_{2^jP}(x)\mathrm{d}x.$$

Note that 2^jP can be covered by $\approx \frac{R}{2^n}$ balls of radius $\approx \frac{2^j2^n}{R}$. Therefore, using Lemma 4.1, we get

$$\|\mu_D\phi_P\|_1 \lesssim \sum_{j=1}^\infty 2^{-\frac{Mj}{2}} 2^{n\alpha-n}R^{1-\alpha} \lesssim 2^{n\alpha-n}R^{1-\alpha}. \tag{22}$$

Using Lemma 4.1 once again, we obtain

$$\|\mu_D\phi_P\|_\infty \lesssim \|\mu_D\|_\infty \lesssim 2^{nd-n\alpha}. \tag{23}$$

Using (23) and (22), we obtain

$$\begin{aligned} \|\mu_D\phi_P\|_{q'} &\leq \|\mu_D\phi_P\|_\infty^{1/q}\|\mu_D\phi_P\|_1^{1/q'} \\ &\lesssim 2^{n\frac{d-\alpha}{q}}(2^{n\alpha-n}R^{1-\alpha})^{1/q'}. \end{aligned} \tag{24}$$

Using (20), (21), (24) and then Cauchy-Schwarz inequality, we get

$$\begin{aligned} \|f_I^\vee f_J^\vee\|_{L^1(\mathrm{d}\mu)} &\lesssim R^{1-\frac{\alpha}{q'}} 2^{n(\alpha(1-\frac{2}{q})+d-2)} \sum_P \|f_{I,P}\|_2\|f_{J,P}\|_2 \\ &\lesssim R^{1-\frac{\alpha}{q'}} 2^{n(\alpha(1-\frac{2}{q})+d-2)} \Big[\sum_P \|f_{I,P}\|_2^2\Big]^{\frac{1}{2}} \Big[\sum_P \|f_{J,P}\|_2^2\Big]^{\frac{1}{2}} \end{aligned}$$

Using the Schwartz decay of ϕ, the fact that the rectangles P tile $\mathbf{R}^d$ and Plancherel formula, we get

$$\|f_I^\vee f_J^\vee\|_{L^1(\mathrm{d}\mu)} \lesssim R^{1-\frac{\alpha}{q'}} 2^{n(\alpha(1-\frac{2}{q})+d-2)} \|f_I\|_2 \|f_J\|_2. \tag{25}$$

The exponent of 2^n in (25) is non-negative and $2^n \lesssim R$. Therefore

$$\begin{aligned} \|f_I^\vee f_J^\vee\|_{L^1(\mathrm{d}\mu)} &\lesssim R^{1-\frac{\alpha}{q'}} R^{\alpha(1-\frac{2}{q})+d-2} \|f_I\|_2 \|f_J\|_2 \\ &\lesssim R^{d-1-\frac{\alpha}{q}} \|f_I\|_2 \|f_J\|_2. \end{aligned} \tag{26}$$

Finally, using (26) and L^2-orthogonality, as in [23] and [25], we bound S_1. Note that for each dyadic cap I, there are finitely many (depending on d) dyadic caps J related to I. Therefore, for each I,

$$\sum_{J \sim I} \|f_J\|_2 \lesssim \|f_{I'}\|_2,$$

for a cap I' of sidelength $C2^n$ which contains I. Also note that for each n, the caps $\{I' : \ell(I) = 2^n\}$ are finitely overlapping. Thus,

$$\sum_{\ell(I)=2^n} \|f_I\|_2^2 \approx \sum_{\ell(I)=2^n} \|f_{I'}\|_2^2 \approx \|f\|_2^2.$$

Therefore,

$$\begin{aligned} \sum_{\ell(I)=2^n,\, I\sim J} \|f_I\|_2 \|f_J\|_2 &\le \Big[\sum_{\ell(I)=2^n} \|f_I\|_2^2\Big]^{1/2} \Big[\sum_{\ell(I)=2^n} \Big(\sum_{J\sim I} \|f_J\|_2\Big)^2\Big]^{1/2} \\ &\lesssim \|f\|_2^2. \end{aligned}$$

Using this, (26) and the fact that there are $\lesssim \log(R)$ values of n in the sum for S_1 in (13), we obtain (for each $q > \frac{d+2}{d}$)

$$S_1 \lesssim R^{d-1-\frac{\alpha}{q}}.$$

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