

DISPERSIVE ESTIMATES FOR SCHRÖDINGER OPERATORS IN THE PRESENCE OF A RESONANCE AND/OR AN EIGENVALUE AT ZERO ENERGY IN DIMENSION THREE: II

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ABSTRACT. We investigate boundedness of the evolution $e^{it\mathcal{H}}$ in the sense of $L^2(\mathbb{R}^3) \rightarrow L^2(\mathbb{R}^3)$ as well as $L^1(\mathbb{R}^3) \rightarrow L^\infty(\mathbb{R}^3)$ for the non-selfadjoint operator

$$\mathcal{H} = \begin{bmatrix} -\Delta + \mu - V_1 & -V_2 \\ V_2 & \Delta - \mu + V_1 \end{bmatrix}$$

where $\mu > 0$ and V_1, V_2 are real-valued decaying potentials. Such operators arise when linearizing a focusing NLS equation around a standing wave and the aforementioned bounds are needed in the study of nonlinear asymptotic stability of such standing waves. We derive our results under some natural spectral assumptions (corresponding to a ground state soliton of NLS), see A1)–A4) below, but without imposing any restrictions on the edges $\pm\mu$ of the essential spectrum. Our goal is to develop an “axiomatic approach”, which frees the linear theory from any nonlinear context in which it may have arisen.

1. THE MATRIX CASE: INTRODUCTION

Consider the Schrödinger operator $H = -\Delta + V$ in \mathbb{R}^3 , where V is a real-valued potential. Let P_{ac} be the orthogonal projection onto the absolutely continuous subspace of $L^2(\mathbb{R}^3)$ which is determined by H . In Journé, Soffer, Sogge [JouSofSog], Yajima [Yaj1], Rodnianski, Schlag [RodSch], Goldberg, Schlag [GolSch] and Goldberg [Gol], $L^1(\mathbb{R}^3) \rightarrow L^\infty(\mathbb{R}^3)$ dispersive estimates for the time evolution $e^{itH}P_{ac}$ were investigated under various decay assumptions on the potential V and the assumption that zero is neither an eigenvalue nor a resonance of H . Recall that zero energy is a resonance iff there is $f \in L^{2,-\sigma}(\mathbb{R}^3) \setminus L^2(\mathbb{R}^3)$ for all $\sigma > \frac{1}{2}$ so that $Hf = 0$. Here $L^{2,-\sigma} = \langle x \rangle^\sigma L^2$ are the usual weighted L^2 spaces and $\langle x \rangle := (1 + |x|^2)^{\frac{1}{2}}$. For a survey of recent work in this area see [Sch2].

In [ErdSch], the authors investigated dispersive estimates when there is a resonance or eigenvalue at energy zero. It is well-known, see Rauch [Rau], Jensen, Kato [JenKat], and Murata [Mur], that the decay in that case is $t^{-\frac{1}{2}}$. Moreover, these authors derived expansions of the evolution into inverse powers of time in weighted $L^2(\mathbb{R}^3)$ spaces. In [ErdSch], the authors obtained such expansions with respect to the $L^1 \rightarrow L^\infty$ norm, albeit only in terms of the powers $t^{-\frac{1}{2}}$ and $t^{-\frac{3}{2}}$. Independently, Yajima [Yaj2] achieved similar results.

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In this paper we obtain analogous expansions for a class of **matrix Schrödinger operators**. Consider the matrix Schrödinger operator

$$\mathcal{H} = \mathcal{H}_0 + V = \begin{bmatrix} -\Delta + \mu & 0 \\ 0 & \Delta - \mu \end{bmatrix} + \begin{bmatrix} -V_1 & -V_2 \\ V_2 & V_1 \end{bmatrix}$$

on $L^2(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$. Here $\mu > 0$ and V_1, V_2 are real-valued. It follows from Weyl's criterion that the essential spectrum of \mathcal{H} is $(-\infty, -\mu] \cup [\mu, \infty)$. The discrete spectrum may intersect $\mathbb{C} \setminus \mathbb{R}$, and the algebraic and geometric multiplicities of eigenvalues may be different (i.e., \mathcal{H} has a nonzero nilpotent part at these eigenvalues).

Such operators appear naturally as linearizations of a nonlinear Schrödinger equation around a standing wave (or *soliton*), see below. Dispersive estimates in the context of such linearizations were obtained in Cuccagna [Cuc], Rodnianski, Schlag, Soffer [RodSchSof1], and [Sch1] under various decay assumptions on the potential and the assumption that zero is neither an eigenvalue nor a resonance of \mathcal{H} . In addition, one always assumes that there are no imbedded eigenvalues in the essential spectrum.

The emphasis of the present paper is to develop an "abstract" (or "axiomatic") approach, which frees the linear theory from any reference to a nonlinear context in which it may have arisen. More specifically, our results will require the following assumptions on \mathcal{H} (in what follows, σ_3 is one of the Pauli matrices, see (14)):

Assumptions:

A1) $-\sigma_3 V$ is a positive matrix

A2) $L_- := -\Delta + \mu - V_1 + V_2 \geq 0$

A3) For some $\beta > 0$,

$$(1) \quad |V_1(x)| + |V_2(x)| \lesssim \langle x \rangle^{-\beta}$$

A4) There are no imbedded eigenvalues in $(-\infty, -\mu) \cup (\mu, \infty)$

Assumptions A1)-A3) hold in the important example of a linearized nonlinear Schrödinger equation, provided the linearization is performed around the (positive) ground state standing wave. Indeed, suppose that $\psi(t, x) = e^{it\alpha^2} \phi(x)$ is a standing wave solution of the NLS

$$i\partial_t \psi + \Delta \psi + |\psi|^{2\beta} \psi = 0,$$

where $\beta > 0$. Here we assume that ϕ is a ground state, i.e.,

$$\alpha^2 \phi - \Delta \phi = \phi^{2\beta+1}, \quad \phi > 0.$$

It is known that such ϕ exist and that they are radial, smooth, and exponentially decaying, see Strauss [Str1], Berestycki, Lions [BerLio1] and for uniqueness, see Coffman [Cof], McLeod, Serin [McLSer], and Kwong [Kwo]. Linearizing around the standing wave solution yields a matrix potential with $V_1 = (\beta + 1)\phi^{2\beta}$ and $V_2 = \beta\phi^{2\beta}$. Hence $V_1 > 0$ and $V_1 > |V_2|$, which is the same as Assumption A1). Moreover, $L_- = -\Delta + \alpha^2 - \phi^{2\beta}$ satisfies $L_- \phi = 0$ and $L_- \geq 0$ follows from $\phi > 0$.

There is a large body of literature concerning the orbital (or Lyapunov) stability (or instability) of this ground state standing wave, see for example Shatah [Sha], Shatah, Strauss [ShaStr], Weinstein [Wei1], [Wei2], Cazenave, Lions [CazLio], Grillakis, Shatah, Strauss [GriShaStr1], [GriShaStr2], and Comech, Pelinovsky [ComPel]. Reviews of much of this work are in Strauss [Str2], and Sulem, Sulem [SulSul].

The question of when the stronger property of asymptotic stability holds has received a lot of attention over the past decade. Starting with Soffer and Weinstein [SofWei1], [SofWei2], who studied the modulation equations governing the evolution of *small* solitons¹, there has been much work also on the case of large solitons, see Buslaev, Perelman [BusPer1], [BusPer2], Cuccagna [Cuc], Perelman [Per1], [Per2], Rodnianski, Soffer, Schlag [RodSchSof1], [RodSchSof2]. It is for this purpose, rather than for the aforementioned orbital stability, that the dispersive estimates of the present paper are of relevance. Let us note that for the case of small solitons the potentials V_1, V_2 will be small and therefore the matrix operator above becomes easier to treat (this is because of dimension three and analogous to the case of scalar Schrödinger operators with small potentials, see e.g., Rodnianski, Schlag [RodSch]). Only for large V_1, V_2 can significant (spectral) difficulties arise on the *linear* level.

It is known that Assumption A2) implies that the spectrum $\text{spec}(\mathcal{H})$ satisfies $\text{spec}(\mathcal{H}) \subset \mathbb{R} \cup i\mathbb{R}$ and that all points of the discrete spectrum other than zero are eigenvalues whose geometric and algebraic multiplicities coincide. For this see Grillakis [Gri], [BusPer1] or [RodSchSof1], as well as Section 2 below.

Unfortunately it is unknown at this point how to guarantee Assumption A4), although it is believed to hold for systems that arise from a ground state soliton as explained above (in 1-d this is known, see Perelman [Per1], due to the explicit form of the ground state in that case). It would be desirable to have an "abstract" approach to this question. But so far this is unknown, and it is an important open problem to settle this issue (even for radial potentials). Note that there can be imbedded eigenvalues for $V_2 = 0$ and V_1 large and positive. But in that case Assumption A2) does not hold. However, Assumptions A2) and A3) alone do not imply A4) by an example of Denisov [Den]. Let us remark that because of these examples where imbedded eigenvalues can exist for our systems even though the potentials are smooth and decay rapidly, it seems certain that the methods known for the scalar case (say, commutator methods in the spirit of Mourre theory) alone will not suffice. Some extra information needs to be used (like A2 plus additional restrictions) to insure the absence of imbedded eigenvalues.

For the case of scalar Schrödinger operators it is widely known that imbedded eigenvalues are unstable. In fact, under generic perturbations they turn into resonances in the complex plane (Fermi golden rule). Hence, one may hope that A4) holds generically in a suitable sense. However, in the matrix case the situation is more complicated and imbedded eigenvalues can turn into complex eigenvalues under small perturbations, see Cuccagna, Pelinovsky, and Vougalter [CucPelVou], [CucPel]. More precisely, whether or not this happens depends on the sign of $\langle \sigma_3 \mathcal{H} f, f \rangle$ where f belongs to the

¹Such solitons only arise in an NLS equation with a linear potential. They are generated by bifurcation off a bound state of the linear Schrödinger operator.

real subspace associated with an imbedded eigenvalue. This is analogous to Krein's theorem and the Krein signature in classical mechanics, see MacKay [MacK], or Avez, Arnold [AveArn].

Unlike the self-adjoint case, for our matrix operators \mathcal{H} the boundedness of $\|e^{it\mathcal{H}}\|_{2 \rightarrow 2}$ as $|t| \rightarrow \infty$ is generally false. Indeed, this is the case in the presence of any complex spectrum. Moreover, even if there is no complex spectrum, then this operator norm can grow polynomially in t due a nonzero nilpotent part of the root-space of \mathcal{H} at zero. Thus, we are lead to consider the boundedness of $\|e^{it\mathcal{H}}P_s\|_{2 \rightarrow 2}$, where $I - P_s$ is the Riesz projection corresponding to the discrete spectrum. This has been studied before in the case where the thresholds $\pm\mu$ are neither eigenvalues nor resonances, see [Cuc, CucPelVou, RodSchSof1]. In fact, the first results on such L^2 (or H^1)-boundedness are due to Weinstein [Wei1], [Wei2] who used variational methods. Such an approach is intimately tied up with the underlying nonlinear problem because it uses the properties of the ground state. For this reason, Weinstein needs to assume that he is in the stable (L^2 -subcritical) case. However, the recent work [Sch1] requires such bounds also in the super-critical case.

Our first result establishes such an L^2 bound in the full generality of Assumptions A1)-A4). In particular, it shows that neither threshold resonances nor threshold eigenvalues affect the L^2 -boundedness.

Theorem 1. *Assume that V satisfies Assumptions A1)-A4) with $\beta > 5$. Then*

$$\sup_{t \in \mathbb{R}} \|e^{it\mathcal{H}}P_s\|_{2 \rightarrow 2} \leq C$$

with a constant that depends on V .

In this context we would like to mention the work of Gesztesy, Jones, Latushkin, and Stanislavova [GesJonLatSta]. They prove, for linearized NLS, that $\sigma(e^{it\mathcal{H}}P_s) = \{z : |z| = 1\}$.

In order to formulate our main dispersive estimate, we need to introduce the analogue of the projection onto the continuous spectrum from the self-adjoint case. This is done as follows. First, let P_d be the Riesz projection corresponding to the discrete spectrum of \mathcal{H} . Second, let P_μ be the projection with range equal to $\ker(\mathcal{H} - \mu)$ and kernel equal to $(\ker(\mathcal{H}^* - \mu))^\perp$. Moreover, $P_\mu = 0$ if μ is not an eigenvalue of \mathcal{H} . Similarly with $P_{-\mu}$. We show below, see Lemma 10, that $P_{\pm\mu}$ are well-defined, and that $P_d, P_\mu, P_{-\mu}$ commute. In fact, $P_d P_\mu = P_d P_{-\mu} = P_\mu P_{-\mu} = 0$. Now, define

$$P_c = (I - P_d)(I - P_\mu)(I - P_{-\mu}) = I - P_d - P_\mu - P_{-\mu}.$$

Clearly, P_c is the analogue of the continuous spectral projection in the self-adjoint case. It eliminates all the eigenfunctions, including those at the thresholds (recall that we are assuming absence of imbedded eigenvalues).

Theorem 2. *Assume that V satisfies Assumptions A1)-A4) with $\beta > 10$. Then there exists a time-dependent operator F_t such that*

$$\sup_t \|F_t\|_{L^1 \rightarrow L^\infty} < \infty, \quad \left\| e^{it\mathcal{H}}P_c - t^{-1/2}F_t \right\|_{1 \rightarrow \infty} \leq Ct^{-3/2}.$$

If both μ and $-\mu$ are not eigenvalues, then F_t is of rank at most two. Moreover, if $\pm\mu$ are neither eigenvalues nor resonances, then $F_t \equiv 0$.

In all cases, the operators F_t can be given explicitly, and they can be extracted from our proofs with more work. We carry this out explicitly for the case when $\pm\mu$ are not eigenvalues, see formula (58) below. For scalar Schrödinger operators, such explicit representations of the kernels of F_t (in terms of resonance functions and projections onto the eigenspaces) were derived by Yaajima [Yaj2]. His formulas show that F_t has finite rank in all cases, and the same should be true in Theorem 2. It is important to note that the $t^{-\frac{3}{2}}$ bound is destroyed by an eigenvalue at zero, even if zero is not a resonance and even after projecting the zero eigenfunction away (this was discovered by Jensen, Kato [JenKat] for scalar operators).

Finally, we remark that it was not our intention to obtain the minimal value of β in Assumption A3). Our results can surely be improved in that regard. Needless to say, the problem of lowering the requirement on β is only one of many remaining issues. More relevant to nonlinear questions seems to be how to prove A4), and/or how to deal with imbedded eigenvalues when they do occur (in regards to our theorems). In a similar vein, it would of course be interesting to develop this linear theory when A2) does not hold. This is the case, for example, when linearizing around excited states, see [BerLio2].

2. THE MATRIX CASE: GENERALITIES

In this section we shall develop some standard and well-known properties of the spectra and resolvents of \mathcal{H} under Assumptions A2)-A4). It should be mentioned that Assumption A1) seems to be needed only in order to apply the symmetric resolvent identity, see Section 3 below. However, in this section we work with the usual resolvent identity and therefore do not need A1).²

Lemma 3. *Let $\beta > 0$ be arbitrary in (1). Then the essential spectrum of \mathcal{H} equals $(-\infty, -\mu] \cup [\mu, \infty)$. Moreover, $\text{spec}(\mathcal{H}) = -\text{spec}(\mathcal{H}) = \overline{\text{spec}(\mathcal{H})} = \text{spec}(\mathcal{H}^*)$ and $\text{spec}(\mathcal{H}) \subset \mathbb{R} \cup i\mathbb{R}$. The discrete spectrum of \mathcal{H} consists of eigenvalues $\{z_j\}_{j=1}^N$, $0 \leq N \leq \infty$, of finite multiplicity. For each $z_j \neq 0$ the algebraic and geometric multiplicities coincide and $\text{Ran}(\mathcal{H} - z_j)$ is closed. The zero eigenvalue has finite algebraic multiplicity, i.e., the generalized eigenspace $\bigcup_{k=1}^{\infty} \ker(\mathcal{H}^k)$ has finite dimension. In fact, there is a finite $m \geq 1$ such that $\ker(\mathcal{H}^k) = \ker(\mathcal{H}^{k+1})$ for all $k \geq m$.*

Proof. The statement about the essential spectrum follows from Weyl's criterium. To see this, note

that conjugation of \mathcal{H} by the matrix $\begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix}$ leads to the matrix operator

$$\begin{bmatrix} 0 & iL_- \\ -iL_+ & 0 \end{bmatrix}$$

where L_- is as above and with $L_+ = -\Delta + \mu - V_1 - V_2$. We will again denote this matrix by \mathcal{H} . Let $H_0 = -\Delta + \mu$ and set $W_1 = -V_1 + V_2$, $W_2 = -V_1 - V_2$,

$$(2) \quad \mathcal{H}_0 = \begin{bmatrix} 0 & iH_0 \\ -iH_0 & 0 \end{bmatrix}, \quad W = \begin{bmatrix} 0 & iW_1 \\ -iW_2 & 0 \end{bmatrix},$$

²It seems that one can work with the usual resolvent identity throughout this paper, which would then allow us to dispense with A1) altogether. However, A1) holds in important applications and we find it convenient to use the symmetric resolvent identity.

$$\mathcal{H} = \mathcal{H}_0 + W = i \begin{bmatrix} 0 & H_0 + W_1 \\ -H_0 - W_2 & 0 \end{bmatrix}.$$

By means of the matrix $J = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$ one can also write

$$\mathcal{H}_0 = \begin{bmatrix} H_0 & 0 \\ 0 & H_0 \end{bmatrix} J, \quad \mathcal{H} = \begin{bmatrix} H_0 + W_1 & 0 \\ 0 & H_0 + W_2 \end{bmatrix} J.$$

Clearly, \mathcal{H} is a closed operator on the domain $\text{Dom}(\mathcal{H}) = W^{2,2} \times W^{2,2}$. Since $\mathcal{H}_0^* = \mathcal{H}_0$ it follows that $\text{spec}(\mathcal{H}_0) \subset \mathbb{R}$. One checks that for $\Re z \neq 0$

$$\begin{aligned} (\mathcal{H}_0 - z)^{-1} &= -(\mathcal{H}_0 + z) \begin{bmatrix} (H_0^2 - z^2)^{-1} & 0 \\ 0 & (H_0^2 - z^2)^{-1} \end{bmatrix} \\ (3) \quad &= - \begin{bmatrix} (H_0^2 - z^2)^{-1} & 0 \\ 0 & (H_0^2 - z^2)^{-1} \end{bmatrix} (\mathcal{H}_0 + z) \end{aligned}$$

$$(4) \quad (\mathcal{H} - z)^{-1} = (\mathcal{H}_0 - z)^{-1} - (\mathcal{H}_0 - z)^{-1} U_1 \left[1 + U_2 J (\mathcal{H}_0 - z)^{-1} U_1 \right]^{-1} U_2 J (\mathcal{H}_0 - z)^{-1}$$

where (4) also requires the expression in brackets to be invertible, and with

$$U_1 = \begin{bmatrix} |W_1|^{\frac{1}{2}} & 0 \\ 0 & |W_2|^{\frac{1}{2}} \end{bmatrix}, \quad U_2 = \begin{bmatrix} |W_1|^{\frac{1}{2}} \text{sign}(W_1) & 0 \\ 0 & |W_2|^{\frac{1}{2}} \text{sign}(W_2) \end{bmatrix}.$$

It follows from (3) that $\text{spec}(\mathcal{H}_0) = (-\infty, -\mu] \cup [\mu, \infty) \subset \mathbb{R}$. Since $V_1(x) \rightarrow 0$ and $V_2(x) \rightarrow 0$ as $x \rightarrow \infty$, it follows from Weyl's theorem, see Theorem XIII.14 in [ReeSim4], and the representation (4) for the resolvent of \mathcal{H} , that $\text{spec}_{\text{ess}}(\mathcal{H}) = \text{spec}_{\text{ess}}(\mathcal{H}_0) = (-\infty, -\mu] \cup [\mu, \infty) \subset \mathbb{R}$. Moreover, (4) implies via the analytic Fredholm alternative that $(\mathcal{H} - z)^{-1}$ is a meromorphic function in $\mathbb{C} \setminus (-\infty, -\mu] \cup [\mu, \infty)$. Furthermore, the poles are eigenvalues³ of \mathcal{H} of finite multiplicity and $\text{Ran}(\mathcal{H} - z_j)$ is closed at each pole z_j .

The symmetries of the spectrum are consequences of the commutation properties of \mathcal{H} with the Pauli matrices

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Now let us check that the spectrum lies in the union of the real and imaginary axes. Thus, suppose that

$$\begin{bmatrix} 0 & iL_- \\ -iL_+ & 0 \end{bmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = E \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

³Note that since \mathcal{H} is not self-adjoint, it can happen that

$$\ker(\mathcal{H} - z)^2 \neq \ker(\mathcal{H} - z)$$

for some $z \in \mathbb{C}$. In other words, \mathcal{H} can possess *generalized eigenspaces*. In the NLS applications this does happen at $z = 0$ due to symmetries like modulation.

with $E \neq 0$ and $\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in L^2 \setminus \{0\}$. Then $f_1 \neq 0$ and $f_2 \neq 0$ and $f_1 \perp \ker\{L_-\}$. Hence, $g = L_-^{-\frac{1}{2}} f_1$ satisfies

$$L_-^{\frac{1}{2}} L_+ L_-^{\frac{1}{2}} g = E^2 g$$

and thus $E^2 \in \mathbb{R}$, as desired. Here we used that $\sqrt{L_-} L_+ \sqrt{L_-}$ with domain $W^{4,2}(\mathbb{R}^3)$ is a selfadjoint operator. For a proof of this see Lemma 11.10 in [RodSchSof2]. That same lemma also contains a proof of the fact that for any eigenvalue other than zero the algebraic and geometric multiplicities coincide.

Let P_0 be the Riesz projection at zero. Then, on the one hand one checks that

$$\text{Ran} P_0 \supset \ker(\mathcal{H}^m) \quad \text{for all } m \geq 1.$$

On the other hand, if $\|(\mathcal{H} - z)^{-1}\| \leq C|z|^{-\nu}$, then

$$\mathcal{H}^\nu P_0 = 0.$$

Thus $\text{Ran} P_0 \subset \ker(\mathcal{H}^\nu)$. See [HisSig] Chapter 6 for these general statements about Riesz projections. \square

It will follow from the next section that $N < \infty$ in Lemma 3 provided $\beta > 5$ (which can probably be relaxed). Indeed, in that section we will derive expansions of the resolvent $(\mathcal{H} - z)^{-1}$ about the thresholds $\pm\mu$ which will preclude the eigenvalues from accumulating at these points. Thus there can only be finitely many eigenvalues, i.e., $N < \infty$.

Next, we need to develop a limiting absorption principle for the resolvents $(\mathcal{H} - z)^{-1}$ when $|z| > \mu$. As observed in [CucPelVou] and [Sch1], this can be done along the lines of the classical Agmon argument [Agm]. For the sake of completeness, we present some of the arguments from [Sch1].

We begin by recalling some weighted L^2 estimates for the free resolvent $(\mathcal{H}_0 - z)^{-1}$ which go by the name "limiting absorption principle". The weighted L^2 -spaces here are the usual ones $L^{2,\sigma} = \langle x \rangle^{-\sigma} L^2$. It will be convenient to introduce the space

$$X_\sigma := L^{2,\sigma}(\mathbb{R}^3) \times L^{2,\sigma}(\mathbb{R}^3).$$

Clearly, $X_\sigma^* = X_{-\sigma}$. The statement is that

$$(5) \quad \sup_{|\lambda| \geq \lambda_0, 0 < \epsilon} |\lambda|^{\frac{1}{2}} \|(\mathcal{H}_0 - (\lambda \pm i\epsilon))^{-1}\|_{X_\sigma \rightarrow X_\sigma^*} < \infty$$

provided $\lambda_0 > \mu$ and $\sigma > \frac{1}{2}$ and was proved in this form by Agmon [Agm]. By the explicit expression for the kernel of the free resolvent in \mathbb{R}^3 one obtains the existence of the limit

$$\lim_{\epsilon \rightarrow 0^+} \langle (\mathcal{H}_0 - (\lambda \pm i\epsilon))^{-1} \phi, \psi \rangle$$

for any $\lambda \in \mathbb{R}$ and any pair of Schwartz functions ϕ, ψ . Hence $(\mathcal{H}_0 - (\lambda \pm i0))^{-1}$ satisfies the same bound as in (5) provided $|\lambda| \geq \lambda_0 > \mu$. There is a corresponding bound which is valid for all energies. It takes the form

$$(6) \quad \sup_{z \in \mathbb{C}} \|(\mathcal{H}_0 - z)^{-1}\|_{X_\sigma \rightarrow X_\sigma^*} < \infty$$

provided $\sigma > 1$. It is much more elementary to obtain than (5) since it only uses that the convolution with $|x|^{-1}$ is bounded from $L^{2,\sigma}(\mathbb{R}^3) \rightarrow L^{2,-\sigma}(\mathbb{R}^3)$ provided $\sigma > 1$. In fact, it is Hilbert-Schmidt in these norms. We now state a lemma about absence of imbedded resonances.

Lemma 4. *Let $\beta > 1$. Then for any $\lambda \in \mathbb{R}$, $|\lambda| > \mu$ the operator $(\mathcal{H}_0 - (\lambda \pm i0))^{-1}V$ is a compact operator on $X_{-\frac{1}{2}-} \rightarrow X_{-\frac{1}{2}-}$ and*

$$I + (\mathcal{H}_0 - (\lambda \pm i0))^{-1}V$$

is invertible on these spaces.

Proof. The compactness is standard and we refer the reader to [Agm] or [ReeSim4]. Let $\lambda > \mu$. By the Fredholm alternative, the invertibility statement requires excluding solutions $(\psi_1, \psi_2) \in X_{-\frac{1}{2}-}$ of the system

$$\begin{aligned} 0 &= \psi_1 - R_0(\lambda - \mu + i0)(V_1\psi_1 + V_2\psi_2) \\ 0 &= \psi_2 - R_0(-\lambda - \mu)(V_2\psi_1 + V_1\psi_2), \end{aligned}$$

where $R_0(z)$ is the free, scalar resolvent $(-\Delta - z)^{-1}$. Notice that these equations imply that $\psi_2 \in L^2$ and that

$$\begin{aligned} 0 &= \langle \psi_1, V_1\psi_1 \rangle + \langle \psi_1, V_2\psi_2 \rangle - \langle R_0(\lambda - \mu + i0)(V_1\psi_1 + V_2\psi_2), V_1\psi_1 + V_2\psi_2 \rangle \\ 0 &= \langle \psi_2, V_2\psi_1 \rangle - \langle R_0(-\lambda - \mu)(V_2\psi_1 + V_1\psi_2), V_2\psi_1 \rangle \\ 0 &= \langle \psi_2, V_1\psi_2 \rangle - \langle R_0(-\lambda - \mu)V_2\psi_1, V_1\psi_2 \rangle - \langle R_0(-\lambda - \mu)V_1\psi_2, V_1\psi_2 \rangle. \end{aligned}$$

Since V_1, V_2 are real-valued, inspection of these equations reveals that

$$\Im \langle R_0(\lambda - \mu + i0)(V_1\psi_1 + V_2\psi_2), V_1\psi_1 + V_2\psi_2 \rangle = 0.$$

So Agmon's well-known bootstrap lemma (see Theorem 3.2 in [Agm]) can be used to conclude that $\psi_1 \in L^2(\mathbb{R}^3)$. But then we have an imbedded eigenvalue at λ , which contradicts Assumption A4). So one can invert

$$I + (\mathcal{H}_0 - (\lambda \pm i0))^{-1}V$$

on $X_{-\frac{1}{2}-}$ and we are done. \square

As usual, one converts the information of the previous lemma into a bound for the perturbed resolvent by means of the resolvent identity.

Proposition 5. *Let $\beta > 1$ and fix an arbitrary $\lambda_0 > \mu$. Then*

$$(7) \quad \sup_{|\lambda| \geq \lambda_0, 0 < \epsilon} |\lambda|^{\frac{1}{2}} \|(\mathcal{H} - (\lambda \pm i\epsilon))^{-1}\| < \infty$$

where the norm is the one from $X_{\frac{1}{2}+} \rightarrow X_{-\frac{1}{2}-}$.

Proof. Let $z = \lambda + i\epsilon$, $\lambda \geq \lambda_0$, $\epsilon \neq 0$. By the resolvent identity and the fact that the spectrum of \mathcal{H} belongs to $\mathbb{R} \cup i\mathbb{R}$,

$$(8) \quad (\mathcal{H} - z)^{-1} = (I + (\mathcal{H}_0 - z)^{-1}V)^{-1}(\mathcal{H}_0 - z)^{-1}$$

as operators on $L^2(\mathbb{R}^3)$. Because of the $|\lambda|^{-\frac{1}{2}}$ -decay in (5), there exists a positive radius r_V such that

$$\|(\mathcal{H}_0 - z)^{-1}V\| < \frac{1}{2}$$

for all $|z| > r_V$ in the operator norm of $X_{-\frac{1}{2}-} \rightarrow X_{-\frac{1}{2}-}$. In conjunction with (8) this implies that

$$\|(\mathcal{H} - z)^{-1}\| \leq C|z|^{-\frac{1}{2}}$$

for all $|z| > r_V$ in the operator norm of $X_{\frac{1}{2}+} \rightarrow X_{-\frac{1}{2}-}$. Now suppose (7) fails. It then follows from (8) and (6) that there exist a sequence z_n with $\Re(z_n) \geq \lambda_0$ and functions $f_n \in X_{-\frac{1}{2}-}$ with $\|f_n\|_{X_{-\frac{1}{2}-}} = 1$ and such that

$$(9) \quad \|[I + (\mathcal{H}_0 - z_n)^{-1}V]f_n\|_{X_{-\frac{1}{2}-}} \rightarrow 0$$

as $n \rightarrow \infty$. Necessarily, the z_n accumulate at some point $\lambda \in [\lambda_0, r_V]$. Without loss of generality, $z_n \rightarrow \lambda$ and $\Im(z_n) > 0$ for all $n \geq 1$. Next, we claim that (9) also holds in the following form:

$$(10) \quad \|[I + (\mathcal{H}_0 - (\lambda + i0))^{-1}V]f_n\|_{X_{-\frac{1}{2}-}} \rightarrow 0$$

as $n \rightarrow \infty$. If so, then it would clearly contradict Lemma 4. To prove (10), let

$$S := I + (\mathcal{H}_0 - (\lambda + i0))^{-1}V$$

for simplicity. Then

$$(11) \quad \begin{aligned} I + (\mathcal{H}_0 - z_n)^{-1}V &= S + ((\mathcal{H}_0 - z_n)^{-1} - (\mathcal{H}_0 - (\lambda + i0))^{-1})V \\ &= [I + ((\mathcal{H}_0 - z_n)^{-1} - (\mathcal{H}_0 - (\lambda + i0))^{-1})VS^{-1}]S. \end{aligned}$$

Our claim now follows from the fact that the expression in brackets is an invertible operator for large n on $X_{-\frac{1}{2}-}$. This in turn relies on bounds of the form: Given $\epsilon > 0$, there exists $\delta > 0$ so that for $\Re z > 0$, and all z' close to z ,

$$(12) \quad \|(-\Delta - z)^{-1} - (-\Delta - z')^{-1}\|_{L^{2, \frac{1}{2} + \epsilon} \rightarrow L^{2, -\frac{1}{2} - \epsilon}} \leq C_{\delta, \epsilon} |z - z'|^\delta$$

see [Agm].⁴ □

As in the case of the free Hamiltonian \mathcal{H}_0 , it is now possible to define the boundary values of the resolvent $(\mathcal{H} - z)^{-1}$. More precisely, the following corollary holds.

Corollary 6. *Let $\beta > 1$. Define*

$$(13) \quad (\mathcal{H} - (\lambda \pm i0))^{-1} := (I + (\mathcal{H}_0 - (\lambda \pm i0))^{-1}V)^{-1}(\mathcal{H}_0 - (\lambda \pm i0))^{-1}$$

for all $|\lambda| > \mu$. Then as $\epsilon \rightarrow 0+$,

$$\|(\mathcal{H} - (\lambda \pm i\epsilon))^{-1} - (\mathcal{H} - (\lambda \pm i0))^{-1}\| \rightarrow 0$$

in the norm of $X_{\frac{1}{2}+} \rightarrow X_{-\frac{1}{2}-}$ and one can extend (7) to $\epsilon \geq 0$.

⁴Of course $\delta \rightarrow 0$ as $\epsilon \rightarrow 0$. Moreover, if $\delta = 1$, then one needs $\epsilon > 1$.

Proof. Definition (13) is legitimate by Lemma 4 and motivated by (8). Thus, the resolvent $(\mathcal{H} - (\lambda \pm i\epsilon))^{-1}$ is well-defined for all $\epsilon \geq 0$ and $|\lambda| > \lambda_0$. In view of (12),

$$\|(\mathcal{H}_0 - (\lambda \pm i\epsilon))^{-1} - (\mathcal{H}_0 - (\lambda \pm i0))^{-1}\| \rightarrow 0$$

as $\epsilon \rightarrow 0$ in the norm of $X_{\frac{1}{2}+} \rightarrow X_{-\frac{1}{2}-}$. Moreover, by (11) and again (12),

$$\begin{aligned} & [I + (\mathcal{H}_0 - (\lambda + i\epsilon))^{-1}V]^{-1} - [I + (\mathcal{H}_0 - (\lambda + i0))^{-1}V]^{-1} \\ &= S^{-1}[I + ((\mathcal{H}_0 - (\lambda + i\epsilon))^{-1} - (\mathcal{H}_0 - (\lambda + i0))^{-1})VS^{-1}]^{-1} - S^{-1} \\ &= \sum_{k=1}^{\infty} S^{-1}[-((\mathcal{H}_0 - (\lambda + i\epsilon))^{-1} - (\mathcal{H}_0 - (\lambda + i0))^{-1})VS^{-1}]^k \end{aligned}$$

tends to zero in the norm of $X_{-\frac{1}{2}-}$ as $\epsilon \rightarrow 0+$. \square

3. RESOLVENT EXPANSIONS AT THRESHOLDS

In view of Assumption A1), we write

$$V = -\sigma_3 v v^* = -\sigma_3 v^2 =: v_1 v_2,$$

where $v_1 = -\sigma_3 v$, $v_2 = v^* = v$,

$$(14) \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

It follows from (1) that the entries of v_1, v_2 are real-valued and decay like $\langle x \rangle^{-\beta/2}$. Let $\lambda = \mu + z^2$, where $\text{Im}(z) > 0$ and $|z|$ small. We have the symmetric resolvent identity:

$$(15) \quad R(\lambda) := (\mathcal{H} - \lambda)^{-1} = R_0(\lambda) - R_0(\lambda)v_1(I + v_2R_0(\lambda)v_1)^{-1}v_2R_0(\lambda).$$

Recall that (see previous section) the essential spectrum of \mathcal{H} is $(-\infty, -\mu] \cup [\mu, \infty)$. As in the scalar case [ErdSch], we obtain resolvent expansions at the threshold $\lambda = \mu$ in the case of a resonance and/or eigenvalue. Recall that $R_0(\lambda)$ has the kernel

$$R_0(\lambda)(x, y) = \frac{1}{4\pi|x-y|} \begin{bmatrix} e^{iz|x-y|} & 0 \\ 0 & -e^{-\sqrt{2\mu+z^2}|x-y|} \end{bmatrix}.$$

We have a similar representation of $R_0(\lambda)$ for λ around $-\mu$. Let

$$\begin{aligned} A(z) &= I + v_2R_0(\lambda)v_1 \\ &=: A_0 + zA_1(z), \end{aligned}$$

where

$$\begin{aligned} A_0 &= I + v_2R_0(\mu)v_1, \\ A_1(z) &=: \frac{1}{z}v_2(R_0(\lambda) - R_0(\mu))v_1, \\ R_0(\mu)(x, y) &= \frac{1}{4\pi|x-y|} \begin{bmatrix} 1 & 0 \\ 0 & -e^{-\sqrt{2\mu}|x-y|} \end{bmatrix}. \end{aligned}$$

If $\beta > 3$, then $v_2R_0(\lambda)v_1$ is a Hilbert-Schmidt operator. Hence, $\ker A_0$ is finite-dimensional.

Lemma 7. [JenNen] *Let $F \subset \mathbb{C} \setminus \{0\}$ have zero as an accumulation point. Let $A(z)$, $z \in F$, be a family of bounded operators of the form*

$$A(z) = A_0 + zA_1(z)$$

with $A_1(z)$ uniformly bounded as $z \rightarrow 0$. Suppose that 0 is an isolated point of the spectrum of A_0 , and let S be the corresponding Riesz projection. Assume that $\text{rank}(S) < \infty$. Then for sufficiently small $z \in F$ the operators

$$B(z) := \frac{1}{z}(S - S(A(z) + S)^{-1}S)$$

are well-defined and bounded on \mathcal{H} . Moreover, if $A_0 = A_0^$, then they are uniformly bounded as $z \rightarrow 0$. The operator $A(z)$ has a bounded inverse in \mathcal{H} if and only if $B(z)$ has a bounded inverse in $S\mathcal{H}$, and in this case*

$$(16) \quad A(z)^{-1} = (A(z) + S)^{-1} + \frac{1}{z}(A(z) + S)^{-1}SB(z)^{-1}S(A(z) + S)^{-1}.$$

See [ErdSch] for the proof.

We use Lemma 7 to obtain an expansion of $A(z)^{-1}$. Assume A_0 is not invertible. Let S_1 be the Riesz projection corresponding to 0. As in the scalar case, A_0 is self adjoint and it is a compact perturbation of the identity. Therefore, $S_1 = P_{\ker A_0}$, $A_0 + S_1$ is invertible and

$$(17) \quad S_1 = (A_0 + S_1)^{-1}S_1 = S_1(A_0 + S_1)^{-1}.$$

Also note that, if V satisfies (1) for some $\beta > 3$, then

$$\sup_{\substack{z \text{ small} \\ \text{Im}(z) \geq 0}} \|A_1(z)\|_{HS} < \infty$$

Thus, $A(z) + S_1$ is invertible for small z . By Lemma 7 we have

$$A(z)^{-1} = (A(z) + S_1)^{-1} + \frac{1}{z}(A(z) + S_1)^{-1}S_1m(z)^{-1}S_1(A(z) + S_1)^{-1},$$

where

$$\begin{aligned} m(z) &= \frac{1}{z}(S_1 - S_1(A(z) + S_1)^{-1}S_1) \\ &= \frac{-1}{z}S_1 \left[\sum_{k=1}^{\infty} (-1)^k z^k (A_1(z)(A_0 + S_1)^{-1})^k \right] S_1 \\ &= S_1 A_1(z) S_1 + \sum_{k=1}^{\infty} (-1)^k z^k S_1 (A_1(z)(A_0 + S_1)^{-1})^{k+1} S_1 \\ &=: S_1 A_1(0) S_1 + z m_1(z). \end{aligned}$$

We used (17) in the second equality. Let $f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$ and define

$$P_1 f := \int_{\mathbb{R}^3} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} f(x) dx \quad \text{and} \quad P_2 f := \int_{\mathbb{R}^3} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} f(x) dx.$$

Note that

$$A_1(0) = \frac{i}{4\pi} v_2 P_1 v_1.$$

Therefore,

$$m(0) = \frac{-i}{4\pi} S_1 v P_1 v S_1.$$

As in the scalar case, if $m(0)$ is invertible, then we invert $m(z)$ using Neumann series. Otherwise, let $S_2 = P_{\ker m(0)} : S_1 L^2 \rightarrow S_1 L^2$. Obviously, $m(z) + S_2$ is invertible for small z . Using Lemma 7, we have

$$m(z)^{-1} = (m(z) + S_2)^{-1} + \frac{1}{z} (m(z) + S_2)^{-1} S_2 b(z)^{-1} S_2 (m(z) + S_2)^{-1},$$

where

$$\begin{aligned} (18) \quad b(z) &= \frac{1}{z} \left(S_2 - S_2 (m(z) + S_2)^{-1} S_2 \right) \\ &= \frac{-1}{z} S_2 \left[\sum_{k=1}^{\infty} (-1)^k z^k (m_1(z)(m(0) + S_2)^{-1})^k \right] S_2 \\ &=: b(0) + z b_1(z), \end{aligned}$$

where

$$b(0) = S_2 m_1(0) S_2.$$

Note that

$$(19) \quad m_1(z) = S_1 \frac{A_1(z) - A_1(0)}{z} S_1 + \sum_{k=1}^{\infty} (-1)^k z^{k-1} S_1 (A_1(z)(A_0 + S_1)^{-1})^{k+1} S_1.$$

Therefore

$$\begin{aligned} (20) \quad b(0) &= S_2 m_1(0) S_2 \\ &= \frac{1}{8\pi} S_2 v_2 \begin{bmatrix} -|x-y| & 0 \\ 0 & \frac{e^{\sqrt{2\mu}|x-y|}}{\sqrt{2\mu}} \end{bmatrix} v_1 S_2 \\ &= \frac{1}{8\pi} S_2 v \begin{bmatrix} |x-y| & 0 \\ 0 & \frac{1}{\sqrt{2\mu}} e^{-\sqrt{2\mu}|x-y|} \end{bmatrix} v S_2. \end{aligned}$$

Below, we will characterize the projections S_1, S_2 and prove that $b(0)$ is always invertible in $S_2 L^2$.

Lemma 8. *Assume $\beta > 3$. Then*

i) $f \in S_1 L^2 \setminus \{0\}$ if and only if $f = v_2 g$ for some $g \in L^{2, -\frac{1}{2}-} \setminus \{0\}$ such that

$$(21) \quad (\mathcal{H}_0 - \mu)g + Vg = 0 \text{ in } \mathcal{S}'.$$

ii) Assume $f \in S_1 L^2 \setminus \{0\}$, then the following are equivalent

- a) $f \in S_2 L^2 \setminus \{0\}$,*
- b) $P_1 v f = 0$,*
- c) $f = v_2 g$ for some $g \in L^2 \setminus \{0\}$ satisfying (21).*

Proof. If $f \in S_1 L^2 \setminus \{0\}$, then

$$A_0 f = f + v_2 R_0(\mu) v_1 f = 0$$

by definition. Hence, $f = v_2 g$ where

$$g = -R_0(\mu) v_1 f \in L^{2, -\frac{1}{2}-}(\mathbb{R}^3)$$

by the mapping properties of $(-\Delta)^{-1}$. Moreover,

$$g + R_0(\mu)Vg = 0.$$

By Lemma 2.4 in [JenKat] this is equivalent with (21). Conversely, if (21) holds, then we set $f = v_2g$ which belongs to L^2 and satisfies

$$f + v_2R_0(\mu)v_1f = 0.$$

Thus, $S_1f = f$, and the first part is proven.

For the second part, suppose that $S_1f = f$. If in addition $S_2f = f$, then $m(0)f = 0$ which is the same as $S_1vP_1vf = 0$. But then also

$$\langle S_1vP_1vf, f \rangle = \langle P_1vf, vf \rangle = \begin{pmatrix} (P_1vf)_1^2 \\ 0 \end{pmatrix}$$

where we have written $P_1vf = \begin{pmatrix} (P_1vf)_1 \\ 0 \end{pmatrix}$. Hence $P_1vf = 0$. This implies that

$$g = -R_0(\mu)v_1f \in L^2(\mathbb{R}^3)$$

This is a standard property, see for example Lemma 6 in [ErdSch]. In view of the first part of this proof $f = v_2g$.

These implications can be reversed: Indeed, if

$$g = -R_0(\mu)v_1f \in L^2(\mathbb{R}^3)$$

then it follows easily that $P_1v_1f = 0$ which is the same as $P_1vf = 0$ (see for example Lemma 6 in [ErdSch]). But then also $m(0)f = 0$, and the lemma follows. \square

Next, we show that the Jensen-Nenciu expansion stops after (at most) two steps.

Lemma 9. *Assume $\beta > 5$. Then, as an operator in S_2L^2 , the kernel of $b(0)$ is trivial.*

Proof. Assume $f \in S_2L^2$ is in $\ker b(0)$. Since $b(0)$ has a real-valued kernel, we can assume that f is real-valued. Let $f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$ and $h = vf = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$. By Lemma 8 ii), we have $\int h_1 = 0$, $h \equiv -\sigma_3Vg$ for some real-valued $g = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \in L^{2, -\frac{1}{2}-}$ satisfying (21) and

$$(22) \quad h_1 = V_1g_1 + V_2g_2, \quad h_2 = V_2g_1 + V_1g_2.$$

Moreover, since $f \in \ker b(0)$ (again by Lemma 8), we have

$$(23) \quad \left\langle h, \begin{bmatrix} |x-y| & 0 \\ 0 & \frac{1}{\sqrt{2\mu}}e^{-\sqrt{2\mu}|x-y|} \end{bmatrix} h \right\rangle = 0.$$

Now use the following fact from [JenKat] (see also the proof of Lemma 7 in [ErdSch]): if $\int u = \int v = 0$, and $u, v \in L^{2,s}$, $s > 5/2$, then

$$\begin{aligned} \langle |x-y|u, v \rangle &= -\frac{1}{2\pi} \left\langle \frac{1}{|x|} * u, \frac{1}{|x|} * v \right\rangle \\ &= -8\pi \langle (-\Delta)^{-1}u, (-\Delta)^{-1}v \rangle. \end{aligned}$$

Thus,

$$(23) = -8\pi \|(-\Delta)^{-1}h_1\|_2^2 + \langle h_2, \frac{1}{\sqrt{2\mu}} e^{-\sqrt{2\mu}|x-y|} h_2 \rangle = 0$$

Define $\hat{f}(\xi) = \int e^{-i\xi \cdot x} f(x) dx$. Recall that (see, e.g., [Ste])

$$\begin{aligned} \frac{\widehat{e^{-\sqrt{2\mu}|x|}}}{4\pi|x|}(\xi) &= (\xi^2 + 2\mu)^{-1}, \\ \frac{\widehat{e^{-\sqrt{2\mu}|x|}}}{\sqrt{2\mu}}(\xi) &= \frac{8\pi}{(\xi^2 + 2\mu)^2}. \end{aligned}$$

Thus,

$$(24) \quad \|(-\Delta)^{-1}h_1\|_2^2 = \|(-\Delta + 2\mu)^{-1}h_2\|_2^2.$$

On the other hand, by (21), we have

$$(25) \quad \begin{bmatrix} -\Delta - V_1 & -V_2 \\ V_2 & \Delta - 2\mu + V_1 \end{bmatrix} \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = 0.$$

Using this and (22), we obtain

$$\begin{aligned} -\Delta g_1 - V_1 g_1 - V_2 g_2 = 0 &\quad \Rightarrow & h_1 = -\Delta g_1, \\ \Delta g_2 - 2\mu g_2 + V_1 g_2 + V_2 g_1 = 0 &\quad \Rightarrow & h_2 = (-\Delta + 2\mu)g_2. \end{aligned}$$

Adding the equalities on the left hand side, we obtain

$$L_-(g_1 - g_2) = (-\Delta + \mu - V_1 + V_2)(g_1 - g_2) = \mu(g_1 + g_2).$$

Pairing this with $g_1 - g_2$, we have (recall that g_1, g_2 are real-valued)

$$\begin{aligned} \langle L_-(g_1 - g_2), g_1 - g_2 \rangle &= \mu (\|g_1\|_2^2 - \|g_2\|_2^2) \\ &= 0 \text{ by (24)} \end{aligned}$$

The positivity assumption $L_- \geq 0$ implies that $\ker L_- = \text{span}\{\varphi\}$ (if $\ker L_- = \{0\}$, then $\varphi = 0$. Otherwise $\varphi \neq 0$). Therefore,

$$g_1 - g_2 = k\varphi, \text{ for some } k \in \mathbb{R}.$$

Using this in (25), we have

$$\begin{aligned} \begin{bmatrix} -\Delta - V_1 & -V_2 \\ V_2 & \Delta - 2\mu + V_1 \end{bmatrix} \begin{pmatrix} g_1 \\ g_1 - k\varphi \end{pmatrix} = 0 &\quad \Rightarrow \\ (-\Delta - V_1 - V_2)g_1 + kV_2\varphi = 0, & \\ (\Delta - 2\mu + V_1 + V_2)g_1 - k(\Delta - 2\mu + V_1)\varphi = 0. & \end{aligned}$$

Adding the last two inequalities and using the fact that $\varphi \in \ker L_-$, we have

$$g_1 = \frac{k}{2}\varphi \quad \Rightarrow \quad g_2 = -\frac{k}{2}\varphi.$$

If $k \neq 0$, we use (25) once more to conclude that

$$\begin{bmatrix} -\Delta - V_1 & -V_2 \\ V_2 & \Delta - 2\mu + V_1 \end{bmatrix} \begin{pmatrix} \varphi \\ -\varphi \end{pmatrix} = 0.$$

This implies that

$$(-\Delta - V_1 + V_2)\varphi = 0 \quad \Rightarrow \quad \mu\varphi = 0 \quad \Rightarrow \quad \varphi \equiv 0$$

Hence, in all cases $g_1 = g_2 = g$. But then

$$\begin{aligned} -\Delta g - V_1 g - V_2 g &= 0 \\ \Delta g - 2\mu g + V_1 g + V_2 g &= 0 \end{aligned}$$

which implies that $\mu g = 0$ and thus also $g = 0$. Retracing our steps we conclude that $h = 0$ and $f = 0$. Therefore, $\ker b(0) = \{0\}$ and we are done. \square

Lemmas 7, 8 and 9 imply that $A(z)$ is always invertible for small $z \neq 0$ and

$$\begin{aligned} (26) \quad A(z)^{-1} &= (A(z) + S_1)^{-1} \\ &+ \frac{1}{z} (A(z) + S_1)^{-1} S_1 (m(z) + S_2)^{-1} (A(z) + S_1)^{-1} + \\ &+ \frac{1}{z^2} (A(z) + S_1)^{-1} S_1 (m(z) + S_2)^{-1} S_2 b(z)^{-1} S_2 (m(z) + S_2)^{-1} S_1 (A(z) + S_1)^{-1}. \end{aligned}$$

Note that

$$A(z)^{-1} = \frac{1}{z^2} S_2 b(0)^{-1} S_2 + O\left(\frac{1}{z}\right).$$

With $\lambda = \mu + z^2$,

$$\begin{aligned} (27) \quad R_V(\lambda) &= R_0(\lambda) - R_0(\lambda) v_1 (A(z))^{-1} v_2 R_0(\lambda) \\ &= -\frac{1}{z^2} R_0(\lambda) \sigma_3 v S_2 b(0)^{-1} S_2 v R_0(\lambda) + \dots \end{aligned}$$

The most singular term in this expansion can be identified as a (not necessarily orthogonal) projection onto the eigenspace at the threshold.

Lemma 10. *Let $\beta > 5$. Then the operator $P_\mu := -R_0(\mu) \sigma_3 v S_2 b(0)^{-1} S_2 v R_0(\mu)$ is a projection in $L^2(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ with the property that*

$$\text{Ran}(P_\mu) = \ker(\mathcal{H} - \mu), \quad \ker(P_\mu) = \ker(\mathcal{H}^* - \mu)^\perp.$$

Proof. Choose a basis $\{\varphi_j\}_{j=1}^r$ of $\ker(\mathcal{H} - \mu)$ so that $\mathcal{B} := \{v\varphi_1, \dots, v\varphi_r\}$ is an orthonormal basis of $\text{Ran}(S_2)$ (which is finite-dimensional). This can be done since v is invertible. Recall that, see (20),

$$S_2 b(0) S_2 = -S_2 v \begin{bmatrix} \frac{-|x-y|}{8\pi} & 0 \\ 0 & \frac{1}{8\pi} \frac{e^{-\sqrt{2\mu}|x-y|}}{\sqrt{2\mu}} \end{bmatrix} \sigma_3 v S_2.$$

We denote the matrix of $S_2 b(0) S_2$ in the basis \mathcal{B} by

$$S_2 b(0) S_2 = \{a_{ij}\}_{i,j=1}^r =: M.$$

Then, with

$$V\varphi_j =: \begin{pmatrix} \alpha_j \\ \beta_j \end{pmatrix},$$

we have

$$\begin{aligned}
a_{ij} &= \langle v\varphi_i, S_2 b(0) S_2 v\varphi_j \rangle \\
&= \int (\alpha_i(x), \beta_i(x)) \begin{bmatrix} \frac{|x-y|}{8\pi} & 0 \\ 0 & \frac{1}{8\pi} \frac{e^{-\sqrt{2\mu}|x-y|}}{\sqrt{2\mu}} \end{bmatrix} \begin{pmatrix} \alpha_j(y) \\ \beta_j(y) \end{pmatrix} dy dx \\
&= \left\langle \alpha_i \frac{|x-y|}{8\pi}, \alpha_j \right\rangle + \left\langle \beta_i \frac{1}{8\pi} \frac{e^{-\sqrt{2\mu}|x-y|}}{\sqrt{2\mu}}, \beta_j \right\rangle \\
&= -\langle (-\Delta)^{-1} \alpha_i, (-\Delta)^{-1} \alpha_j \rangle + \langle (-\Delta + 2\mu)^{-1} \beta_i, (-\Delta + 2\mu)^{-1} \beta_j \rangle \\
&= -\left\langle \sigma_3 R_0(\mu) \begin{pmatrix} \alpha_i \\ \beta_i \end{pmatrix}, R_0(\mu) \begin{pmatrix} \alpha_j \\ \beta_j \end{pmatrix} \right\rangle = -\langle \sigma_3 R_0(\mu) V\varphi_i, R_0(\mu) V\varphi_j \rangle \\
(28) \quad &= -\langle \sigma_3 \varphi_i, \varphi_j \rangle.
\end{aligned}$$

Here we used that

$$R_0(\mu) = \begin{bmatrix} -\Delta & 0 \\ 0 & \Delta - 2\mu \end{bmatrix}^{-1} = \begin{bmatrix} (-\Delta)^{-1} & 0 \\ 0 & (\Delta - 2\mu)^{-1} \end{bmatrix}.$$

Next, write $S_2 v R_0(\mu)$ relative to the orthonormal basis \mathcal{B} :

$$\begin{aligned}
S_2 v R_0(\mu) f &= \sum_j \langle v R_0(\mu) f, v\varphi_j \rangle v\varphi_j \\
&= -\sum_j \langle f, \sigma_3 R_0(\mu) V\varphi_j \rangle v\varphi_j \\
&= \sum_j \langle f, \sigma_3 \varphi_j \rangle v\varphi_j.
\end{aligned}$$

Hence,

$$\begin{aligned}
P_\mu f &= -R_0(\mu) \sigma_3 v S_2 (S_2 b(0) S_2)^{-1} S_2 v R_0(\mu) f \\
&= -R_0(\mu) \sigma_3 v S_2 \sum_{i,j} v\varphi_i M_{ij}^{-1} \langle f, \sigma_3 \varphi_j \rangle \\
&= \sum_{i,j} R_0(\mu) V\varphi_i M_{ij}^{-1} \langle f, \sigma_3 \varphi_j \rangle \\
(29) \quad &= -\sum_{i,j} \varphi_i M_{ij}^{-1} \langle f, \sigma_3 \varphi_j \rangle.
\end{aligned}$$

The following properties hold:

- i) $\text{Ran} P_\mu \subseteq \ker(\mathcal{H} - \mu) = \text{span}\{\varphi_1, \dots, \varphi_r\}$
- ii) $P_\mu \varphi_k = \varphi_k$
- iii) $\ker P_\mu = \ker(\mathcal{H}^* - \mu)^\perp = \text{span}\{\sigma_3 \varphi_1, \dots, \sigma_3 \varphi_r\}^\perp$

Property *i*) is immediate from (29), whereas *ii*) follows from (28) and (29):

$$\begin{aligned} P_\mu \varphi_k &= - \sum_{i,j} \varphi_i M_{ij}^{-1} \langle \varphi_k, \sigma_3 \varphi_j \rangle = \sum_{i,j} \varphi_i M_{ij}^{-1} a_{j,k} \\ &= \sum_i \varphi_i \delta_{i,k} = \varphi_k. \end{aligned}$$

Finally, property *iii*) can be seen as follows:

$$\begin{aligned} P_\mu f = 0 &\Leftrightarrow \langle f, \sigma_3 \varphi_j \rangle = 0 \text{ for each } j \\ &\Leftrightarrow f \in \text{span}\{\sigma_3 \varphi_1, \dots, \sigma_3 \varphi_r\}^\perp. \end{aligned}$$

Since $\mathcal{H}^* = \sigma_3 \mathcal{H} \sigma_3$, we see that

$$\ker(\mathcal{H}^* - \mu) = \text{span}\{\sigma_3 \varphi_1, \dots, \sigma_3 \varphi_r\}.$$

The lemma follows. \square

Analogously, one obtains expansions around $-\mu$ which involve $P_{-\mu}$. The previous proposition proves that there is a direct – but not orthogonal – sum representation

$$L^2 \times L^2 = \ker(\mathcal{H} - \mu) \dot{+} [\ker(\mathcal{H}^* - \mu)]^\perp$$

as well as

$$L^2 \times L^2 = \ker(\mathcal{H} + \mu) \dot{+} [\ker(\mathcal{H}^* + \mu)]^\perp.$$

Similarly, for any point z in the discrete spectrum

$$L^2 \times L^2 = \ker(\mathcal{H} - z) \dot{+} [\ker(\mathcal{H}^* - z)]^\perp.$$

Finally, it is a simple matter to check the following:

Lemma 11. *The pair-wise products of P_d, P_μ and $P_{-\mu}$ vanish where P_d is the Riesz projection*

$$P_d = -\frac{1}{2\pi i} \oint_\gamma (\mathcal{H} - z)^{-1} dz$$

with a simple closed contour γ surrounding the discrete spectrum.

Proof. Suppose that $\mathcal{H}f = \mu f$. Then

$$P_d f = \frac{1}{2\pi i} \oint_\gamma (z - \mu)^{-1} dz f = 0$$

Hence $P_d P_\mu = 0$. Next, suppose that $\mathcal{H}f = zf$ and $\mathcal{H}^*g = \mu g$ where $z \in \mathbb{C}$ belongs to the discrete spectrum of \mathcal{H} . Then

$$z \langle f, g \rangle = \langle \mathcal{H}f, g \rangle = \langle f, \mathcal{H}^*g \rangle = \mu \langle f, g \rangle$$

which implies that $\langle f, g \rangle = 0$. Consequently,

$$\text{Ran} P_d \subset [\ker(\mathcal{H}^* - \mu)]^\perp = \ker P_\mu$$

and thus $P_\mu P_d = 0$. The same argument also shows that

$$P_\mu P_{-\mu} = P_{-\mu} P_\mu = P_d P_{-\mu} = P_{-\mu} P_d = 0$$

and we are done. \square

4. A SPECTRAL REPRESENTATION OF THE EVOLUTION AND L^2 BOUNDS

The following lemma develops a representation of the (non-unitary) flow $e^{it\mathcal{H}}$ via the resolvents. It relies heavily on the limiting absorption principle from Section 2. The statement is of course analogous to the scalar Schrödinger case in which it is a consequence of the spectral theorem and asymptotic completeness.

Lemma 12. *Under our assumptions A1)-A4) with $\beta > 5$ there is the representation*

$$(30) \quad \begin{aligned} e^{it\mathcal{H}} &= \frac{1}{2\pi i} \int_{|\lambda| \geq \mu} e^{it\lambda} [(\mathcal{H} - (\lambda + i0))^{-1} - (\mathcal{H} - (\lambda - i0))^{-1}] d\lambda \\ &+ \sum_j e^{it\mathcal{H}} P_{\zeta_j} + e^{it\mu} P_{\mu} + e^{-it\mu} P_{-\mu}, \end{aligned}$$

where the sum runs over the entire discrete spectrum $\{\zeta_j\}_j$ and P_{ζ_j} is the Riesz projection corresponding to the eigenvalue ζ_j , whereas $P_{\pm\mu}$ are as above. The formula (30) and the convergence of the integral are to be understood in the following weak sense: If ϕ, ψ belong to $[W^{2,2} \times W^{2,2}(\mathbb{R}^3)] \cap X_{1+}$, then

$$\begin{aligned} \langle e^{it\mathcal{H}} \phi, \psi \rangle &= \lim_{R \rightarrow \infty} \frac{1}{2\pi i} \int_{R \geq |\lambda| \geq \mu} e^{it\lambda} \langle [(\mathcal{H} - (\lambda + i0))^{-1} - (\mathcal{H} - (\lambda - i0))^{-1}] \phi, \psi \rangle d\lambda \\ &+ \sum_j \langle e^{it\mathcal{H}} P_{\zeta_j} \phi, \psi \rangle + e^{it\mu} \langle P_{\mu} \phi, \psi \rangle + e^{-it\mu} \langle P_{-\mu} \phi, \psi \rangle \end{aligned}$$

for all t , where the integrand is well-defined by the limiting absorption principle.

Proof. The evolution $e^{it\mathcal{H}}$ is defined via the Hille-Yoshida theorem. Indeed, let $a > 0$ be large. Then $i\mathcal{H} - a$ satisfies (with ρ the resolvent set)

$$\rho(i\mathcal{H} - a) \supset (0, \infty) \quad \text{and} \quad \|(i\mathcal{H} - a - \lambda)^{-1}\| \leq |\lambda|^{-1} \quad \text{for all } \lambda > 0.$$

The inequality can be seen as follows:

$$\begin{aligned} \|(i\mathcal{H} - (a + \lambda))^{-1}\| &= \|(i\mathcal{H}_0 - (a + \lambda))^{-1} (I + iV(i\mathcal{H}_0 - (a + \lambda))^{-1})^{-1}\| \\ &\leq \frac{1}{a + \lambda} \sum_{k=0}^{\infty} \left(\frac{C}{a + \lambda} \right)^k \leq \frac{1}{a - C + \lambda} \leq \frac{1}{\lambda}, \end{aligned}$$

as claimed. Hence $\{e^{t(i\mathcal{H} - a)}\}_{t \geq 0}$ is a contractive semigroup, so that $\|e^{it\mathcal{H}}\|_{2 \rightarrow 2} \leq e^{|t|a}$ for all $t \in \mathbb{R}$. If $\Re z > a$, then there is the Laplace transform

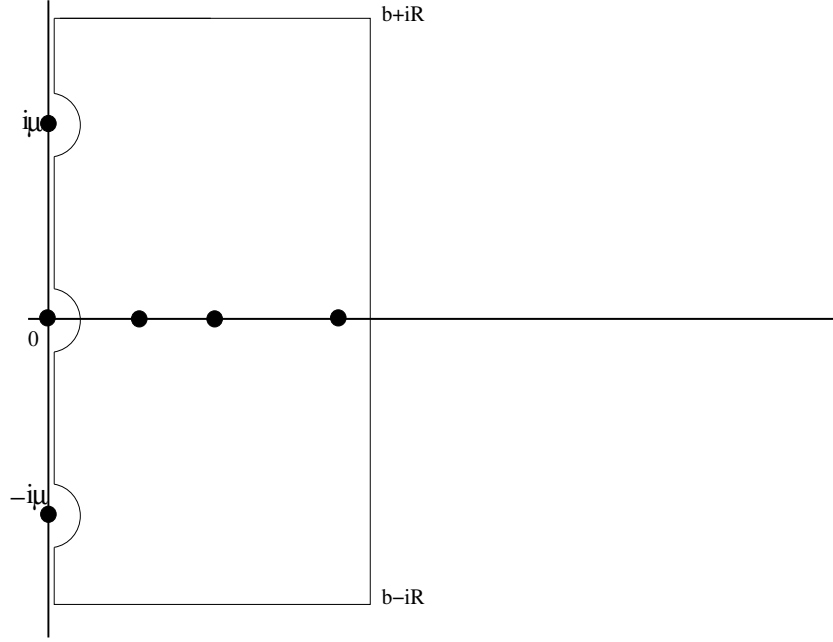
$$(31) \quad (i\mathcal{H} - z)^{-1} = - \int_0^{\infty} e^{-tz} e^{it\mathcal{H}} dt$$

as well as its inverse (with $b > a$ and $t > 0$)

$$(32) \quad e^{it\mathcal{H}} = - \frac{1}{2\pi i} \int_{b-i\infty}^{b+i\infty} e^{tz} (i\mathcal{H} - z)^{-1} dz.$$

While (31) converges in the norm sense, defining (32) requires more care. The claim is that for any $\phi, \psi \in \text{Dom}(\mathcal{H}) = W^{2,2} \times W^{2,2}$,

$$(33) \quad \langle e^{it\mathcal{H}} \phi, \psi \rangle = - \lim_{R \rightarrow \infty} \frac{1}{2\pi i} \int_{b-iR}^{b+iR} e^{tz} \langle (i\mathcal{H} - z)^{-1} \phi, \psi \rangle dz.$$

FIGURE 1. The contour $\Gamma_{R,\delta}^+$

To verify this, let $t > 0$ and use (31) to conclude that

$$\begin{aligned}
 -\frac{1}{2\pi i} \int_{b-iR}^{b+iR} e^{tz} \langle (i\mathcal{H} - z)^{-1} \phi, \psi \rangle dz &= \frac{1}{2\pi i} \int_{b-iR}^{b+iR} e^{tz} \int_0^\infty e^{-sz} \langle e^{is\mathcal{H}} \phi, \psi \rangle ds dz \\
 (34) \qquad \qquad \qquad &= \frac{1}{\pi} \int_0^\infty e^{(t-s)b} \frac{\sin((t-s)R)}{t-s} \langle e^{is\mathcal{H}} \phi, \psi \rangle ds.
 \end{aligned}$$

Since $e^{(t-s)b} \langle e^{is\mathcal{H}} \phi, \psi \rangle$ is a C^1 function in s (recall $\phi \in \text{Dom}(\mathcal{H})$) as well as exponentially decaying in s (because of $b > a$), it follows from standard properties of the Dirichlet kernel that the limit in (34) exists and equals $\langle e^{it\mathcal{H}} \phi, \psi \rangle$, as claimed. Note that if $t < 0$, then the limit is zero. Therefore, it follows that for any $b > a$,

$$\begin{aligned}
 (35) \quad &\langle e^{it\mathcal{H}} \phi, \psi \rangle \\
 &= -\lim_{R \rightarrow \infty} \left\{ \frac{1}{2\pi i} \int_{b-iR}^{b+iR} e^{tz} \langle (i\mathcal{H} - z)^{-1} \phi, \psi \rangle dz - \frac{1}{2\pi i} \int_{-b-iR}^{-b+iR} e^{tz} \langle (i\mathcal{H} - z)^{-1} \phi, \psi \rangle dz \right\}
 \end{aligned}$$

Consider the contour $\Gamma_{R,\delta}^+$ which is depicted in Figure 4. It has the segment $b - iR$ to $b + iR$ as its right boundary, and the left boundary contains semi-circular arcs of radius $\delta > 0$ centered at each imaginary eigenvalue⁵ as well as two semi-circles centered at $\pm i\mu$. Otherwise, the left boundary abuts

⁵This figure depicts the contour for the case where zero is the only point on the imaginary axis that belongs to the discrete spectrum of $i\mathcal{H}$. Generally speaking, semi-circles surround each point of the discrete spectrum on the imaginary axis, as well as the edges $\pm i\mu$.

on the imaginary axis. Now fix R and some small $\delta > 0$ and conclude from the Cauchy theorem that

$$\frac{1}{2\pi i} \oint_{\Gamma_{R,\delta}^+} e^{tz} \langle (i\mathcal{H} - z)^{-1} \phi, \psi \rangle dz = \sum_j \frac{1}{2\pi i} \oint_{\gamma_j} e^{tz} \langle (i\mathcal{H} - z)^{-1} \phi, \psi \rangle dz,$$

where γ_j are small circles (say, of radius δ) around the positive eigenvalues $\{\lambda_j\}_j$ of $i\mathcal{H}$ (in the figure these are indicated by the three dots on the real axis). Recall that the Riesz projection

$$P_{\lambda_j} = \frac{1}{2\pi i} \oint_{\gamma_j} (i\mathcal{H} - z)^{-1} dz$$

satisfies

$$\text{Ran}(P_{\lambda_j}) = \bigcup_{m=1}^{\infty} \ker [(i\mathcal{H} - \lambda_j)^m]$$

and that the right-hand side stabilizes at some finite (minimal) $M_j = M(\lambda_j)$. I.e.,

$$\text{Ran}(P_{\lambda_j}) = \ker [(i\mathcal{H} - \lambda_j)^{M_j}].$$

This is also the minimal M_j with the property that⁶

$$(36) \quad \|(i\mathcal{H} - (z - \lambda_j))^{-1}\| \leq C |z - \lambda_j|^{-M_j}$$

as $z \rightarrow \lambda_j$. Now let $p_j(w)$ be the Taylor polynomial of e^w of degree j , i.e.,

$$|e^w - p_j(w)| \leq C |w|^{j+1}.$$

Then

$$(37) \quad \begin{aligned} \frac{1}{2\pi i} \oint_{\gamma_j} e^{tz} (i\mathcal{H} - z)^{-1} dz &= \frac{1}{2\pi i} \oint_{\gamma_j} e^{t\lambda_j} (e^{t(z-\lambda_j)} - p_j(t(z-\lambda_j))) (i\mathcal{H} - z)^{-1} dz \\ &\quad + \frac{1}{2\pi i} \oint_{\gamma_j} e^{t\lambda_j} p_j(t(z-i\mathcal{H} + i\mathcal{H} - \lambda_j)) (i\mathcal{H} - z)^{-1} dz \\ &= \frac{1}{2\pi i} \oint_{\gamma_j} e^{t\lambda_j} p_j(t(i\mathcal{H} - \lambda_j)) (i\mathcal{H} - z)^{-1} dz \\ &= e^{it\mathcal{H}} \frac{1}{2\pi i} \oint_{\gamma_j} (i\mathcal{H} - z)^{-1} dz = e^{it\mathcal{H}} P_{\lambda_j}. \end{aligned}$$

The integral on the right-hand side of (37) is zero by (36). In conclusion, if we let $\Gamma_{R,\delta}^-$ be the reflection of $\Gamma_{R,\delta}^+$ about the imaginary axis $i\mathbb{R}$, then

$$\begin{aligned} &\frac{1}{2\pi i} \oint_{\Gamma_{R,\delta}^+} e^{tz} (i\mathcal{H} - z)^{-1} dz + \frac{1}{2\pi i} \oint_{\Gamma_{R,\delta}^-} e^{tz} (i\mathcal{H} - z)^{-1} dz \\ &= \sum_{j:\lambda_j \neq 0} \frac{1}{2\pi i} \oint_{\gamma_j} e^{tz} (i\mathcal{H} - z)^{-1} dz = \sum_{j:\lambda_j \neq 0} e^{it\mathcal{H}} P_{\lambda_j}. \end{aligned}$$

Note that by the limiting absorption estimate

$$\lim_{R \rightarrow \infty} \frac{1}{2\pi i} \int_{[iR, b+iR]} e^{tz} \langle (i\mathcal{H} - z)^{-1} \phi, \psi \rangle dz = 0,$$

⁶Under our positivity assumption it follows that the only eigenvalue for which $M_j > 1$ is $\lambda = 0$. Nevertheless, we still present this argument in general, since we also want it to apply to $\lambda = 0$.

as well as

$$\lim_{R \rightarrow \infty} \frac{1}{2\pi i} \int_{[-iR, b-iR]} e^{tz} \langle (i\mathcal{H} - z)^{-1} \phi, \psi \rangle dz = 0.$$

Now let c_δ^+ be a contour that is given as follows: Take a straight line $is + \epsilon$ with $\mu + \delta \leq s < \infty$, then make a circular loop of radius δ centered at $i\mu$, followed by a straight line $is - \epsilon$ with the same s as before. Now pass to the limit $\epsilon \rightarrow 0$. Similarly with c_δ^- . Hence, in view of (35) and the preceding,

$$\begin{aligned} \langle e^{it\mathcal{H}} \phi, \psi \rangle &= \frac{1}{2\pi i} \int_{c_\delta^+} e^{tz} \langle (i\mathcal{H} - z)^{-1} \phi, \psi \rangle dz + \frac{1}{2\pi i} \int_{c_\delta^-} e^{tz} \langle (i\mathcal{H} - z)^{-1} \phi, \psi \rangle dz \\ &\quad + \sum_j \langle e^{it\mathcal{H}} P_{\zeta_j} \phi, \psi \rangle \\ &= \frac{1}{2\pi i} \int_{|\lambda| \geq \mu + \delta} e^{it\lambda} [\langle (\mathcal{H} - (\lambda + i0))^{-1} \phi, \psi \rangle - \langle (\mathcal{H} - (\lambda - i0))^{-1} \phi, \psi \rangle] d\lambda \\ (38) \quad &\quad + \sum_j \langle e^{it\mathcal{H}} P_{\zeta_j} \phi, \psi \rangle + e^{it\mu} \langle P_\mu \phi, \psi \rangle + e^{-it\mu} \langle P_{-\mu} \phi, \psi \rangle + O(\sqrt{\delta}) \end{aligned}$$

where the sum extends over the entire discrete spectrum $\{\zeta_j\}_j$ of $i\mathcal{H}$. The integrals over infinite intervals are to be interpreted in the principal value sense. To pass to (38), we use the asymptotic expansion of the resolvent $(i\mathcal{H} - z)^{-1}$ around $\pm i\mu$. Indeed, by (27) and Lemma 10 the expansion of $(i\mathcal{H} - z)^{-1}$ around $z = i\mu$ is of the form $(i\mu - z)^{-1} P_\mu + O(\sqrt{\delta})$. Sending $\delta \rightarrow 0$ implies the lemma. \square

Remark 1. The sum over ζ_j in (30) takes the form

$$\sum_{\zeta_j} e^{it\mathcal{H}} P_{\zeta_j} = \sum_{\zeta_j \neq 0} e^{it\zeta_j} P_{\zeta_j} + \sum_{k=0}^m \frac{(it)^k}{k!} \mathcal{H}^k P_0$$

where m is the minimal positive integer with $\ker(\mathcal{H}^m) = \ker(\mathcal{H}^{m+1})$. This is due to the fact that all points ζ_j in the discrete spectrum other than zero have the property that $\ker(\mathcal{H} - \zeta_j) = \ker[(\mathcal{H} - \zeta_j)^2]$. This typically fails for $\zeta_j = 0$.

The previous proposition has the following corollary.

Corollary 13. *Under our assumptions A1)-A4) the following stability bound holds:*

$$\sup_{t \geq 0} \|e^{-it\mathcal{H}} P_s f\|_2 \leq C \|f\|_2,$$

where $I - P_s$ is the Riesz-projection corresponding to the discrete spectrum.

Proof. Write $P_s = P_s^+ + P_s^-$, where \pm refers to the positive and negative halves of the essential spectrum, respectively. I.e.,

$$P_s^+ = \frac{1}{2\pi i} \int_{\Gamma} (\mathcal{H} - \lambda)^{-1} d\lambda$$

with the usual "thermometer" shaped contour surrounding $[\mu, \infty)$. Then it suffices to prove

$$\sup_{t \geq 0} \|e^{-it\mathcal{H}} P_s^+ f\|_2 \leq C \|f\|_2.$$

Mainly for clarity of exposition we divide the proof into three cases, namely $S_1 = 0$, $S_1 \neq 0, S_2 = 0$, and finally $S_2 \neq 0$. These operators refer to those arising in the expansion of the resolvent around $\lambda = \mu$. The first case $S_1 = 0$ is what is meant by μ being neither an eigenvalue nor a resonance.

Case 1: $S_1 = 0$

This case has been treated before, and is a consequence of Kato theory. To see this, write

$$e^{-it\mathcal{H}}P_s = \frac{1}{2\pi i} \int_{|\lambda| \geq \mu} e^{it\lambda} [(\mathcal{H} - (\lambda + i0))^{-1} - (\mathcal{H} - (\lambda - i0))^{-1}] d\lambda$$

as explained in the previous lemma. By the symmetric resolvent identity (15)

$$\begin{aligned} (\mathcal{H} - (\lambda \pm i0))^{-1} &= (\mathcal{H}_0 - (\lambda \pm i0))^{-1} \\ &\quad - (\mathcal{H}_0 - (\lambda \pm i0))^{-1} v_1 (I + v_2 (\mathcal{H}_0 - (\lambda \pm i0))^{-1} v_1)^{-1} v_2 (\mathcal{H}_0 - (\lambda \pm i0))^{-1}. \end{aligned}$$

It suffices to show that

$$\int \left| \langle (\mathcal{H}_0 - (\lambda \pm i0))^{-1} v_1 (I + v_2 (\mathcal{H}_0 - (\lambda \pm i0))^{-1} v_1)^{-1} v_2 (\mathcal{H}_0 - (\lambda \pm i0))^{-1} f, g \rangle \right| d\lambda \leq C \|f\|_2 \|g\|_2$$

By our assumption on $\pm\mu$,

$$\sup_{\lambda} \|(I + v_2 (\mathcal{H}_0 - (\lambda \pm i0))^{-1} v_1)^{-1}\|_{2 \rightarrow 2} < \infty$$

Hence, it suffices to show that

$$\int \left| \langle v_2 (\mathcal{H}_0 - (\lambda \pm i0))^{-1} f, v_1^* (\mathcal{H}_0 - (\lambda \mp i0))^{-1} g \rangle \right| d\lambda \leq C \|f\|_2 \|g\|_2.$$

However, by Kato's smoothing theory

$$\int \|v_2 (\mathcal{H}_0 - (\lambda \pm i0))^{-1} f\|_2^2 d\lambda \leq C \|f\|_2^2$$

and similarly for g .

Case 2: $S_1 \neq 0, S_2 = 0$

As in the scalar (self-adjoint) case, we do not expect that a resonance can destroy the L^2 bound.

We need to check again that

$$\sup_{t \geq 0} \left\| \int_{\lambda \geq \mu} e^{it\lambda} [(\mathcal{H} - (\lambda + i0))^{-1} - (\mathcal{H} - (\lambda - i0))^{-1}] d\lambda \right\|_{2 \rightarrow 2} \leq C.$$

By the Kato theory argument in the previous case

$$\sup_{t \geq 0} \left\| \int_{\lambda \geq \mu'} e^{it\lambda} [(\mathcal{H} - (\lambda + i0))^{-1} - (\mathcal{H} - (\lambda - i0))^{-1}] d\lambda \right\|_{2 \rightarrow 2} \leq C(\mu')$$

for any fixed $\mu' > \mu$. It remains to deal with the integral over a small interval $\mu' \geq \lambda \geq \mu$. Since $S_2 = 0$ and $S_1 \neq 0$,

$$(39) \quad \mathcal{R}(z) = -\frac{1}{z} \mathcal{R}_0(z) v_1 S v_2 \mathcal{R}_0(z) + \mathcal{R}_0(z) - \mathcal{R}_0(z) v_1 E(z) v_2 \mathcal{R}_0(z)$$

with a uniformly L^2 -bounded $E(z)$ for small z , where we have set $\lambda = \mu + z^2$ and

$$\begin{aligned} \mathcal{R}(z) &= (\mathcal{H} - (\mu + z^2 + i0))^{-1}, \text{ for } z > 0, \\ \mathcal{R}(z) &= (\mathcal{H} - (\mu + z^2 - i0))^{-1} = \overline{(\mathcal{H} - (\mu + z^2 + i0))^{-1}}, \text{ for } z < 0 \end{aligned}$$

and similarly for \mathcal{R}_0 . The second term on the right-hand side of (39) is just the free evolution and thus is bounded in time. Moreover, the third term can be treated by Kato smoothing since $\|E(z)\|_{2 \rightarrow 2}$ is bounded for small z . We have reduced ourselves to showing that

$$(40) \quad \sup_{t \geq 0} \left\| \int e^{itz^2} \chi(z) \mathcal{R}_0(z) v_1 S v_2 \mathcal{R}_0(z) dz \right\|_{2 \rightarrow 2} \leq C.$$

Here χ is a smooth bump function supported in $[-1, 1]$. Now let M_j for $1 \leq j \leq 2$ be the matrices

$$M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Denote

$$U(t) := \int e^{itz^2} \chi(z) \mathcal{R}_0(z) v_1 S v_2 \mathcal{R}_0(z) dz.$$

Then $M_1 U(t) M_1$ has the kernel

$$(41) \quad \int_{\mathbb{R}^7} \frac{e^{itz^2} e^{iz(|x-u_1|+|y-u_2|)} \chi(z)}{|x-u_1||y-u_2|} r_1(u_1) c_1(u_2) du_1 du_2 dz$$

where r_1 and c_1 are in $L^1(\mathbb{R}^3)$. We claim that

$$(42) \quad \sup_{t \geq 0} \|M_1 U(t) M_1\|_{2 \rightarrow 2} < \infty.$$

In view of (41) this reduces to showing that

$$\sup_{t \geq 0, u_1, u_2} \left| \int_{\mathbb{R}^7} \frac{e^{itz^2} e^{iz(|x-u_1|+|y-u_2|)} \chi(z)}{|x-u_1||y-u_2|} f(x) g(y) dx dy dz \right| \leq C \|f\|_2 \|g\|_2$$

for all $f, g \in L^2(\mathbb{R}^3)$. This in turn is the same as

$$(43) \quad \sup_{t \geq 0} \left| \int_{\mathbb{R}^7} \frac{e^{itz^2} e^{iz(|x|+|y|)} \chi(z)}{|x||y|} f(x) g(y) dx dy dz \right| \leq C \|f\|_2 \|g\|_2.$$

Let

$$F(r) = \chi_{[0, \infty)}(r) r \int_{S^2} f(r\omega) \sigma(d\omega), \quad G(r) = \chi_{[0, \infty)}(r) r \int_{S^2} g(r\omega) \sigma(d\omega).$$

By Cauchy-Schwarz,

$$\|F\|_2 \leq C \|f\|_{L^2(\mathbb{R}^3)}, \quad \|G\|_2 \leq C \|g\|_{L^2(\mathbb{R}^3)}.$$

The following calculation finishes the proof of (42):

$$(43) = \sup_{t \geq 0} \left| \int_{\mathbb{R}^3} e^{itz^2} e^{iz(r+s)} \chi(z) F(r) G(s) dr ds dz \right| \\ = \sup_{t \geq 0} \left| \int_{\mathbb{R}} e^{itz^2} \chi(z) \widehat{F}(z) \widehat{G}(z) dz \right| \\ \leq \|\widehat{F}\|_{L^2(\mathbb{R})} \|\widehat{G}\|_{L^2(\mathbb{R})} = \|F\|_{L^2(\mathbb{R})} \|G\|_{L^2(\mathbb{R})} \\ \leq C \|f\|_2 \|g\|_2.$$

Another contribution is given by $M_2 U(t) M_2$. The kernel here takes the form

$$\int_{\mathbb{R}^7} \frac{e^{itz^2}}{|x-u_1||y-u_2|} \chi(z) e^{-\sqrt{2\mu+z^2}(|x-u_1|+|y-u_2|)} r_2(u_1) c_2(u_2) du_1 du_2 dz$$

where r_2 and c_2 are in $L^2(\mathbb{R}^3)$. The uniform L^2 bound here is even easier, since

$$\left\| \int_{\mathbb{R}^3} \frac{e^{-\sqrt{2\mu+z^2}|x-u|}}{|x-u|} f(u) du \right\|_2 \leq C(1+|z|)^{-1} \|f\|_2$$

so that the desired L^2 bound follows by putting L^2 norms inside the integral.

Finally, we claim that both $\|M_1 U(t) M_2\|_{2 \rightarrow 2}$ and $\|M_2 U(t) M_1\|_{2 \rightarrow 2}$ are bounded in t . Without loss of generality we consider the former.

We first remark that

$$\sup_{t \geq 0} \left\| \int e^{itz^2} \chi(z) M_1 \mathcal{R}_0(z) v_1 S v_2 (\mathcal{R}_0(z) - \mathcal{R}_0(0)) M_2 dz \right\|_{2 \rightarrow 2} < \infty$$

by Kato smoothing theory. Indeed, it is easy to check that

$$\|\chi(z)(\mathcal{R}_0(z) - \mathcal{R}_0(0)) M_2\|_{2 \rightarrow 2} \leq C \begin{cases} |z| & \text{if } |z| < 1 \\ |z|^{-1} & \text{if } |z| > 1 \end{cases}$$

Thus we regain the z which we lost due to the singularity of $\frac{1}{z} S$. It remains to show that

$$\sup_{t \geq 0} \left\| \int e^{itz^2} \chi(z) M_1 \mathcal{R}_0(z) v_1 S v_2 \mathcal{R}_0(0) M_2 dz \right\|_{2 \rightarrow 2} \leq C.$$

This follows from

$$(44) \quad \left| \int_{\mathbb{R}^4} e^{itz^2} \frac{e^{iz|x|}}{|x|} \chi(z) f(x) dx dz \right| \leq C \|f\|_2.$$

As before (with $F(r) = \chi_{[0, \infty)}(r) r \int_{S^2} f(r\omega) \sigma(d\omega)$),

$$(44) = \left| \int_{\mathbb{R}} e^{itz^2} \chi(z) \widehat{F}(z) dz \right| \\ \leq \|\chi\|_2 \|\widehat{F}\|_2 \leq C \|F\|_2 \leq C \|f\|_2$$

and we are done with Case 2.

Case 3: $S_2 \neq 0$

Let $\Gamma_1(z) = (A(z) + S_1)^{-1}$ and $\Gamma_2(z) = (m(z) + S_2)^{-1}$. We proved in the previous section that these are analytic functions for small z and, moreover, for all small $z \neq 0$

$$A(z)^{-1} = \Gamma_1(z) + \frac{1}{z} \Gamma_1(z) S_1 \Gamma_2(z) S_1 \Gamma_1(z) \\ + \frac{1}{z^2} \Gamma_1(z) S_1 \Gamma_2(z) S_2 b(z)^{-1} S_2 \Gamma_2(z) S_1 \Gamma_1(z)$$

see (26). As usual, let $\lambda = \mu + z^2$ with $\Im z > 0$. Then by the symmetric resolvent identity

$$(45) \quad R(\lambda) := (\mathcal{H} - \lambda)^{-1} = R_0(\lambda) - R_0(\lambda) v_1 A(z)^{-1} v_2 R_0(\lambda) \\ = R_0(\lambda) - R_0(\lambda) v_1 \Gamma_1(z) v_2 R_0(\lambda) - \frac{1}{z} R_0(\lambda) v_1 \Gamma_1(z) S_1 \Gamma_2(z) S_1 \Gamma_1(z) v_2 R_0(\lambda)$$

$$(46) \quad - \frac{1}{z^2} R_0(\lambda) v_1 \Gamma_1(z) S_1 \Gamma_2(z) S_2 b(z)^{-1} S_2 \Gamma_2(z) S_1 \Gamma_1(z) v_2 R_0(\lambda)$$

provided z is also small. Note that the contributions of the terms in (45) to the L^2 operator norm has been dealt with in Case 2. Therefore, it suffices to deal with (46). First, set

$$T(z) = v_1 \Gamma_1(z) S_1 \Gamma_2(z) S_2 b(z)^{-1} S_2 \Gamma_2(z) S_1 \Gamma_1(z) v_2$$

and write

$$(47) \quad \begin{aligned} & \frac{1}{z^2} R_0(\lambda) v_1 \Gamma_1(z) S_1 \Gamma_2(z) S_2 b(z)^{-1} S_2 \Gamma_2(z) S_1 \Gamma_1(z) v_2 R_0(\lambda) \\ &= \frac{1}{z} R_0(\lambda) \frac{T(z) - T(0)}{z} R_0(\lambda) + \frac{1}{z^2} R_0(\lambda) T(0) R_0(\lambda) \end{aligned}$$

The first term in (47) only has a z^{-1} singularity, and can therefore be treated as in Case 2. In view of Lemma 10,

$$\begin{aligned} & R_0(\lambda) T(z) R_0(\lambda) \Big|_{z=0} \\ &= R_0(\lambda) v_1 \Gamma_1(z) S_1 \Gamma_2(z) S_2 b(z)^{-1} S_2 \Gamma_2(z) S_1 \Gamma_1(z) v_2 R_0(\lambda) \Big|_{z=0} = P \end{aligned}$$

with P being the projection onto the eigenspace at μ . Now use the resolvent identity again to conclude that (with $\Im z > 0$)

$$(48) \quad R_0(\lambda) T(0) R_0(\lambda) = P - z^2 R_0(\lambda) P - z^2 P R_0(\lambda) + z^4 R_0(\lambda) P R_0(\lambda)$$

Hence, the contribution of the second term in (47) to the L^2 operator norm of $e^{it\mathcal{H}}$ reduces to understanding the operator norm of

$$\int e^{itz^2} z^3 \chi(z) \mathcal{R}_0(z) P \mathcal{R}_0(z) dz$$

with $\mathcal{R}_0(z)$ as above (the first three terms on the right-hand side of (48) are straightforward to deal with). We again need to consider each of the (essentially scalar) operators

$$\int e^{itz^2} z^3 \chi(z) \mathcal{R}_0(z) M_j P M_k \mathcal{R}_0(z) dz$$

for $j, k = 1, 2$ separately. According to (29)

$$P = - \sum_{i,j} \varphi_i M_{ij}^{-1} \langle \cdot, \sigma_3 \varphi_j \rangle$$

where $\{\varphi_j\}_{j=1}^r$ is a suitable basis of $\ker(\mathcal{H} - \mu)$ and M_{ij}^{-1} are some matrix coefficients, see the proof of Lemma 10. Therefore, the case $j = k = 2$ is obvious. The case $j = k = 1$ reduces to establishing that for any f, g which are the first components of functions in $\ker(\mathcal{H} - \mu)$, we have

$$\sup_{t \geq 0} \left\| \int_{\mathbb{R}^7} e^{itz^2} z^3 \chi(z) \frac{e^{iz(|x-x_1|+|y_1-y|)}}{|x-x_1||y_1-y|} f(x_1) g(y_1) dx_1 dy_1 dz \right\|_{2 \rightarrow 2} \leq C$$

or by duality that

$$\sup_{t \geq 0} \left| \int_{\mathbb{R}^{13}} e^{itz^2} z^3 \chi(z) \frac{e^{iz(|x-x_1|+|y_1-y|)}}{|x-x_1||y_1-y|} f(x_1) g(y_1) dx_1 dy_1 dz \phi(x) \psi(y) dx dy \right| \leq C \|\phi\|_2 \|\psi\|_2$$

for any pair ϕ, ψ of Schwartz functions, say. This is the same as showing that

$$\sup_{t \geq 0} \left| \int_{\mathbb{R}^7} e^{itz^2} z^3 \chi(z) \frac{e^{iz(|x|+|y|)}}{|x||y|} (f * \phi)(x) (g * \psi)(y) dx dy dz \right| \leq C \|\phi\|_2 \|\psi\|_2$$

We remark that this estimate is different from the ones we encountered in Case 2 since $f, g \in L^2$ but not necessarily $f, g \in L^1$. We set

$$F(r) = r\chi_{[r>0]} \int_{S^2} (f * \phi)(r\omega) \sigma(d\omega), \quad G(r) = r\chi_{[r>0]} \int_{S^2} (g * \psi)(r\omega) \sigma(d\omega).$$

Since $\phi, \psi \in L^1(\mathbb{R}^3)$, we conclude that $F, G \in L^2_r$. Moreover, $\partial_r F, \partial_r G \in L^2$ and $\widehat{\partial_r F}(z) = z\widehat{F}(z)$, $\widehat{\partial_r G}(z) = z\widehat{G}(z)$. To see that $\partial_r F \in L^2$, observe that

$$\|\partial_r F\|_2 \leq C(\|\nabla(f * \phi)\|_2 + \||x|^{-1}(f * \phi)\|_2) \leq C\|f * \nabla\phi\|_2 \leq C\|f\|_2\|\nabla\phi\|_1$$

where we applied Hardy's inequality in the second step. Hence, we need to show that

$$\sup_{t \geq 0} \left| \int e^{itz^2} z\chi(z)\widehat{\partial_r F}(z)\widehat{\partial_r G}(z) dz \right| \leq C\|\phi\|_2\|\psi\|_2$$

which in turn reduces to proving that

$$\|\partial_r F\|_2 \leq C\|\phi\|_2, \quad \|\partial_r G\|_2 \leq C\|\psi\|_2.$$

For this it suffices to check that

$$\|\nabla(f * \phi)\|_2 + \||x|^{-1}(f * \phi)\|_2 \leq C\|\phi\|_2.$$

By Hardy's inequality the second term on the left-hand side is controlled by the first. Hence, we need to show that

$$\|\widehat{\nabla}f\|_\infty \leq C.$$

We recall that f is assumed to satisfy $f \in L^2(\mathbb{R}^3)$ and

$$-\Delta f = V_1 f + V_2 \tilde{f}$$

for some $\tilde{f} \in L^2(\mathbb{R}^3)$. By the assumed decay of V_1, V_2 we have $V_1 f + V_2 \tilde{f} \in L^1(\mathbb{R}^3)$ and also

$$\int_{\mathbb{R}^3} (V_1 f + V_2 \tilde{f}) dx = 0$$

see Lemma 8. Set $h = V_1 f + V_2 \tilde{f}$. It follows that $|\hat{h}(\xi)| \leq C|\xi|$ and thus $\nabla f = \nabla G_0 h$ implies that

$$\sup_{\xi \in \mathbb{R}^3} |\widehat{\nabla}f(\xi)| \leq C,$$

as desired.

It remains to consider the case $j = 1, k = 2$ ($j = 2, k = 1$ being symmetric). We need to show that for any Schwartz functions ϕ, ψ

$$\sup_{t \geq 0} \left| \int_{\mathbb{R}^7} e^{itz^2} z^3 \chi(z) \frac{e^{iz|x| - |y|\sqrt{2\mu+z^2}}}{|x||y|} (f * \phi)(x)(g * \psi)(y) dx dy dz \right| \leq C\|\phi\|_2\|\psi\|_2$$

where f, g are the first and second components, respectively, of functions in $\ker(\mathcal{H} - \mu)$. Since

$$\int_{\mathbb{R}^3} \frac{e^{-|y|\sqrt{2\mu+z^2}}}{|y|} |(g * \psi)(y)| dy \leq C\|g * \psi\|_\infty \leq C\|g\|_2\|\psi\|_2,$$

we see that we are reduced to showing that

$$\int z^2 \chi(z) |\widehat{\partial_r F}(z)| dz \leq C\|\phi\|_2.$$

This, however, was already established for the case $j = k = 1$, and we are done. \square

5. $L^1 \rightarrow L^\infty$ ESTIMATES

In this section, we bound the $L^1 \rightarrow L^\infty$ operator norm of $e^{it\mathcal{H}}P_s = e^{it\mathcal{H}}P_s^+ + e^{it\mathcal{H}}P_s^-$. In view of the previous section, the kernel of $e^{it\mathcal{H}}P_s^+$, truncated to energies close to μ , is

$$K_t := \frac{1}{2\pi i} \int_{\lambda > \mu} e^{it\lambda} \chi(\lambda - \mu) [(\mathcal{H} - (\lambda + i0))^{-1} - (\mathcal{H} - (\lambda - i0))^{-1}] d\lambda,$$

where χ is an even Schwartz function supported in $(-\lambda_0, \lambda_0)$ and identically equal to 1 in $(\lambda_0/2, \lambda_0/2)$. Here λ_0 is a small constant which will be determined later. By simple change of variables and redefining χ , we have

$$(49) \quad K_t = \frac{1}{\pi i} e^{it\mu} \int_{-\infty}^{\infty} e^{itz^2} z \chi(z) \mathcal{R}(z) dz,$$

where

$$\begin{aligned} \mathcal{R}(z) &= (\mathcal{H} - (\mu + z^2 + i0))^{-1}, \text{ for } z > 0, \\ \mathcal{R}(z) &= (\mathcal{H} - (\mu + z^2 - i0))^{-1} = \overline{(\mathcal{H} - (\mu + z^2 + i0))^{-1}}, \text{ for } z < 0. \end{aligned}$$

We also define

$$\begin{aligned} \mathcal{R}_0(z) &= R_0(\mu + z^2), \text{ for } z > 0, \\ \mathcal{R}_0(z) &= \overline{R_0(\mu + z^2)}, \text{ for } z < 0. \end{aligned}$$

Note that, with this definition, for all $z \in \mathbb{R}$ we have

$$\mathcal{R}_0(z)(x, y) = \frac{1}{4\pi|x-y|} \begin{bmatrix} e^{iz|x-y|} & 0 \\ 0 & -e^{-\sqrt{2\mu+z^2}|x-y|} \end{bmatrix}$$

and

$$(50) \quad \mathcal{R}(z) = \mathcal{R}_0(z) - \mathcal{R}_0(z) v_1 A(z)^{-1} v_2 \mathcal{R}_0(z),$$

where

$$A(z) = I + v_2 \mathcal{R}_0(z) v_1.$$

We will use the following simple lemma repeatedly. It is used in [RodSch, GolSch, ErdSch].

Lemma 14. *Let $F \in L^1$ be differentiable on \mathbb{R} with $F' \in L^1$. Then*

$$\begin{aligned} i) \quad & \left| \int_{-\infty}^{\infty} e^{itz^2} F(z) dz \right| \lesssim t^{-1/2} \|\widehat{F}\|_1, \\ ii) \quad & \left| \int_{-\infty}^{\infty} e^{itz^2} z F(z) dz \right| \lesssim t^{-3/2} \|\widehat{F'}\|_1. \end{aligned}$$

Proof. This follows from

$$|\widehat{e^{itz^2}}(u)| = c|t|^{-\frac{1}{2}}$$

and Parseval's identity. \square

5.1. μ is a resonance but not an eigenvalue. Now, we prove Theorem 2 when μ is a resonance but not an eigenvalue. In this case $S_2 = 0$ and we have

$$A(z)^{-1} = (A(z) + S_1)^{-1} + \frac{1}{z} (A(z) + S_1)^{-1} S_1 m(z)^{-1} S_1 (A(z) + S_1)^{-1},$$

where

$$\begin{aligned} (A(z) + S_1)^{-1} &= (A_0 + S_1)^{-1} + \sum_{k=1}^{\infty} (-1)^k z^k (A_0 + S_1)^{-1} [A_1(z)(A_0 + S_1)^{-1}]^k \\ &=: (A_0 + S_1)^{-1} + zE_1(z), \\ (51) \quad m(z)^{-1} &= m(0)^{-1} + \sum_{k=1}^{\infty} (-1)^k z^k m(0)^{-1} [m_1(z)m(0)^{-1}]^k \\ &=: m(0)^{-1} + zE_2(z). \end{aligned}$$

Thus, using (17), we obtain⁷

$$\begin{aligned} (52) \quad A(z)^{-1} &= \frac{1}{z} S_1 m(0)^{-1} S_1 \\ &\quad + (A(z) + S_1)^{-1} \\ &\quad + E_1(z) S_1 m(z)^{-1} S_1 (A(z) + S_1)^{-1} \\ &\quad + S_1 E_2(z) S_1 (A(z) + S_1)^{-1} \\ &\quad + S_1 m(0)^{-1} S_1 E_1(z) \\ &=: \frac{1}{z} S + E(z). \end{aligned}$$

Note that $m(0) = -\frac{i}{4\pi} S_1 v P_1 v S_1$ is invertible in $S_1 L^2$. Since P_1 is of rank one, both $m(0)$ and S_1 are rank one operators. Let $S_1(x, y) = \varphi(x)\varphi^*(y)$, where φ is the unique function satisfying i) $\|\varphi\|_2 = 1$, ii) $\varphi = v_2 g = v g$ for a resonance function g , as well as iii) $P_1 v \varphi = \binom{c}{0}$ with $c > 0$ (see Lemma 8).

Using c in the definition of $m(0)$, it is easy to see that⁸

$$(53) \quad S(x, y) = \frac{4\pi i}{c^2} S_1(x, y) = \frac{4\pi i}{c^2} \varphi(x)\varphi^*(y).$$

Plugging (52) and (53) into (50), we have

$$\begin{aligned} (54) \quad \mathcal{R}(z) &= -\frac{4\pi i}{c^2 z} \mathcal{R}_0(z) v_1 \varphi \varphi^* v_2 \mathcal{R}_0(z) \\ &\quad + \mathcal{R}_0(z) - \mathcal{R}_0(z) v_1 E(z) v_2 \mathcal{R}_0(z) \\ &= \frac{4\pi i}{c^2 z} \mathcal{R}_0(z) \sigma_3 \psi \psi^* \mathcal{R}_0(z) \\ &\quad + \mathcal{R}_0(z) - \mathcal{R}_0(z) v_1 E(z) v_2 \mathcal{R}_0(z), \end{aligned}$$

where $\psi = v\varphi = -\sigma_3 V g \in L^{2, \beta - \frac{1}{2}-} \subset L^1 \cap L^2(\mathbb{R}^3)$. Using this in (49), we get

$$K_t(x, y) = \frac{1}{\pi i} e^{it\mu} \left(\frac{4\pi i}{c^2} K_1(x, y) + K_2(x, y) - K_3(x, y) \right),$$

⁷We note that the corresponding equation (22) in [ErdSch] has a couple of misprints. It should be replaced with the equation (52). The rest of the proof in [ErdSch] is not affected by this change.

⁸ φ is real-valued, so $\varphi^* = \varphi^t$

where

$$\begin{aligned}
K_1(x, y) &= \int_{-\infty}^{\infty} e^{itz^2} \chi(z) [\mathcal{R}_0(z) \sigma_3 \psi \psi^* \mathcal{R}_0(z)](x, y) dz, \\
K_2(x, y) &= \int_{-\infty}^{\infty} e^{itz^2} z \chi(z) \mathcal{R}_0(z)(x, y) dz \\
(55) \quad K_3(x, y) &= \int_{-\infty}^{\infty} e^{itz^2} z \chi(z) [\mathcal{R}_0(z) v_1 E(z) v_2 \mathcal{R}_0(z)](x, y) dz.
\end{aligned}$$

First, we deal with K_1 . Let $\psi = \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}$.

$$(56) \quad K_1(x, y) = \frac{1}{16\pi^2} \begin{bmatrix} K_1^{11} & K_1^{12} \\ K_1^{21} & K_1^{22} \end{bmatrix},$$

where

$$\begin{aligned}
K_1^{11} &= \int_{\mathbb{R}^7} \frac{e^{itz^2} \chi(z)}{|x - u_1| |y - u_2|} e^{iz(|x - u_1| + |y - u_2|)} \psi_1(u_1) \psi_1(u_2) du_1 du_2 dz \\
K_1^{12} &= - \int_{\mathbb{R}^7} \frac{e^{itz^2} \chi(z)}{|x - u_1| |y - u_2|} e^{iz|x - u_1| - \sqrt{2\mu + z^2}|y - u_2|} \psi_1(u_1) \psi_2(u_2) du_1 du_2 dz \\
K_1^{21} &= \int_{\mathbb{R}^7} \frac{e^{itz^2} \chi(z)}{|x - u_1| |y - u_2|} e^{iz|y - u_2| - \sqrt{2\mu + z^2}|x - u_1|} \psi_2(u_1) \psi_1(u_2) du_1 du_2 dz \\
K_1^{22} &= - \int_{\mathbb{R}^7} \frac{e^{itz^2} \chi(z)}{|x - u_1| |y - u_2|} e^{-\sqrt{2\mu + z^2}(|x - u_1| + |y - u_2|)} \psi_2(u_1) \psi_2(u_2) du_1 du_2 dz.
\end{aligned}$$

As in the scalar case, we will prove that K_1 is a sum of two operators the first one is of finite rank and its $L^1 \rightarrow L^\infty$ norm decays like $t^{-1/2}$, the second one is dispersive, i.e., its $L^1 \rightarrow L^\infty$ norm decays like $t^{-3/2}$. It suffices to prove this claim for each of the components of K_1 .

First we consider K_1^{11} . Let $a_1 = |x - u_1|$, $a_2 = |y - u_2|$ and $a = a_1 + a_2$.

$$K_1^{11} = \int_{\mathbb{R}^7} e^{itz^2} \chi(z) \frac{\cos(za)}{a_1 a_2} \psi_1(u_1) \psi_1(u_2) du_1 du_2 dz.$$

We have (see [ErdSch])

$$\begin{aligned}
(57) \quad \int_{-\infty}^{\infty} e^{itz^2} \chi(z) \cos(za) dz &= \frac{e^{i\pi/4}}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-i(u^2 + a^2)/4t} \widehat{\chi}(u) du \\
&\quad + \frac{e^{i\pi/4}}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-i(u^2 + a^2)/4t} \left(\cos\left(\frac{ua}{2t}\right) - 1 \right) \widehat{\chi}(u) du \\
&= \frac{e^{i\pi/4}}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-i(u^2 + a_1^2 + a_2^2)/4t} \widehat{\chi}(u) du \\
&\quad + \frac{e^{i\pi/4}}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-iu^2/4t} (e^{-ia^2/4t} - e^{-i(a_1^2 + a_2^2)/4t}) \widehat{\chi}(u) du \\
&\quad + \frac{e^{i\pi/4}}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-i(u^2 + a^2)/4t} \left(\cos\left(\frac{ua}{2t}\right) - 1 \right) \widehat{\chi}(u) du \\
&=: C_1 + C_2 + C_3.
\end{aligned}$$

In [ErdSch], we proved that the contribution of C_2 and C_3 in K_1^{11} are dispersive, see pages 367-369 in that paper. The contribution of C_1 is

$$t^{-1/2}F_{11}(t) := \frac{h(t)}{\sqrt{t}} \left[\int_{\mathbb{R}^3} \frac{e^{-i|x-u_1|^2/4t}\psi_1(u_1)}{|x-u_1|} du_1 \right] \left[\int_{\mathbb{R}^3} \frac{e^{-i|y-u_2|^2/4t}\psi_1(u_2)}{|y-u_2|} du_2 \right],$$

where $h(t) = \frac{e^{i\pi/4}}{\sqrt{4\pi}} \int_{-\infty}^{\infty} e^{-iu^2/4t} \widehat{\chi}(u) du$.

Now, we consider K_1^{22} , the others can be treated similarly. Let $a = |x-u_1| + |y-u_2|$.

$$\begin{aligned} K_1^{22}(x, y) &= - \int_{-\infty}^{\infty} \int_{\mathbb{R}^6} \frac{e^{itz^2} \chi(z)}{|x-u_1||y-u_2|} e^{-a\sqrt{2\mu}} \psi_2(u_1) \psi_2(u_2) du_1 du_2 dz \\ &\quad - \int_{-\infty}^{\infty} \int_{\mathbb{R}^6} \frac{e^{itz^2} za\chi(z)}{|x-u_1||y-u_2|} \frac{e^{-a\sqrt{2\mu+z^2}} - e^{-a\sqrt{2\mu}}}{za} \psi_2(u_1) \psi_2(u_2) du_1 du_2 dz \\ &=: t^{-1/2}F_{22}(x, y) + K_{122}(x, y). \end{aligned}$$

By Lemma 14, $\|F_{22}\|_{1 \rightarrow \infty} \lesssim 1$. Before we prove that K_{122} is dispersive, we note that the kernel of the operator F_t in Theorem 2 when μ is a resonance but not an eigenvalue is

$$\begin{aligned} (58) \quad F_t(x, y) &= \frac{e^{it\mu} h(t)}{4\pi^2 c^2} \begin{bmatrix} T_t(\psi_1)(x) T_t(\psi_1)(y) & -T_t(\psi_1)(x) Q(\psi_2)(y) \\ Q(\psi_2)(x) T_t(\psi_1)(y) & -Q(\psi_2)(x) Q(\psi_2)(y) \end{bmatrix} \\ &= \frac{e^{it\mu} h(t)}{4\pi^2 c^2} \begin{bmatrix} T_t(\psi_1)(x) \\ Q(\psi_2)(x) \end{bmatrix} \begin{bmatrix} T_t(\psi_1)(y) & -Q(\psi_2)(y) \end{bmatrix}, \end{aligned}$$

where

$$T_t(f)(x) := \int \frac{e^{-i|x-u|^2/4t} f(u)}{|x-u|} du, \quad Q(f)(x) := \int \frac{e^{-\sqrt{2\mu}|x-u|} f(u)}{|x-u|} du.$$

To prove that K_{122} is dispersive we need the following calculus lemma.

Lemma 15. *For any $k \in \mathbb{R}$ define*

$$g_k(x) = \frac{x}{\sqrt{k^2 + x^2}} e^{-\sqrt{x^2 + k^2}}.$$

Then

$$\|g_k\|_1 + \|g'_k\|_1 + |k| \|g''_k\|_1 + |k|^2 \|g'''_k\|_1 \leq CP(k) e^{-|k|},$$

where P is a polynomial in k .

Proof. Clearly,

$$g_k(x) = -\frac{d}{dx} e^{-\sqrt{x^2 + k^2}}.$$

Hence

$$\|g_k\|_1 = 2 \int_0^{\infty} g_k(x) dx = 2e^{-|k|}.$$

Next, $g_k(x) = \widetilde{g}_k(x/k)$ where

$$\widetilde{g}_k(x) = \frac{x}{\sqrt{1+x^2}} e^{-|k|\sqrt{1+x^2}}.$$

Hence,

$$(59) \quad \|g_k^{(j)}\|_1 = |k|^{-(j-1)} \|\widetilde{g}_k^{(j)}\|_1, \quad j = 1, 2, \dots$$

Note that all derivatives of $\sqrt{1+x^2}$ are bounded functions. Therefore, by Leibnitz's formula

$$|\tilde{g}_k^{(j)}(x)| \lesssim \left| \left(\frac{x}{\sqrt{1+x^2}} \right)^{(j)} \right| e^{-|k|} + |k|(1+|k|^{(j-1)})e^{-|k|\sqrt{1+x^2}}.$$

Note that all derivatives of $\frac{x}{\sqrt{1+x^2}}$ are in L^1 . Thus,

$$\|\tilde{g}_k^{(j)}\|_1 \lesssim e^{-|k|} + (1+|k|^{(j-1)})\| |k|e^{-|k|\sqrt{1+x^2}} \|_1 \lesssim P_j(k)e^{-|k|},$$

where $P_j(k)$ is a polynomial. Using this in (59) yields the assertion of the lemma. \square

Let $h_a(z) := \frac{z}{\sqrt{2\mu a^2 + z^2}} e^{-\sqrt{2\mu a^2 + z^2}}$. In view of Lemma 15 we have the following bounds for h_a :

$$\begin{aligned} \|h_a\|_1 + \|h'_a\|_1 &\lesssim e^{-|a|\sqrt{\mu}}, \\ \|h''_a\|_1 &\lesssim \frac{e^{-|a|\sqrt{\mu}}}{\sqrt{\mu}|a|}, \\ \|h'''_a\|_1 &\lesssim \frac{e^{-|a|\sqrt{\mu}}}{\mu|a|^2}. \end{aligned}$$

Therefore, we have

$$(60) \quad |\widehat{h}_a(\eta)| \lesssim e^{-\mu|a|} \min\left(\frac{1}{\langle \eta \rangle}, \frac{1}{|a|\langle \eta \rangle^2}, \frac{1}{a^2\langle \eta \rangle^3}\right).$$

Note that

$$K_{122}(x, y) = - \int_{-\infty}^{\infty} \int_{\mathbb{R}^6} \int_0^1 \frac{e^{itz} z a \chi(z)}{|x-u_1||y-u_2|} h_a(abz) \psi_2(u_1) \psi_2(u_2) db du_1 du_2 dz.$$

Lemma 14 implies that K_{122} is dispersive if we can prove that

$$(61) \quad \sup_{x, y} \int_{\mathbb{R}^6} \int_0^1 \|(\chi h_a(\widehat{ab(\cdot)}))'\|_1 \frac{a \psi_2(u_1) \psi_2(u_2)}{|x-u_1||y-u_2|} db du_1 du_2$$

is finite. Using the Schwartz decay of $\widehat{\chi}$ and (60), we obtain

$$\begin{aligned} \|(\chi h_a(\widehat{ab(\cdot)}))'\|_1 &= 2\pi \int |\xi| \left| \int \widehat{\chi}(\xi - ba\eta) \widehat{h}_a(\eta) d\eta \right| d\xi \\ &\lesssim \int (1 + |ba\eta|) |\widehat{h}_a(\eta)| d\eta \\ &\lesssim \frac{1}{a} e^{-\sqrt{\mu}a}. \end{aligned}$$

Using this in (61), we have

$$(61) \lesssim \sup_{x, y} \int_{\mathbb{R}^6} e^{-\sqrt{\mu}a} \frac{\psi_2(u_1) \psi_2(u_2)}{|x-u_1||y-u_2|} du_1 du_2 < \infty.$$

This finishes the analysis of K_1 . Note that K_2 is the low energy part of the free evolution and hence it is dispersive. Now, we consider K_3 . Let I_1 be the first coordinate projection with the matrix $I_1 = [1 \ 0]$ and $I_2 = [0 \ 1]$ the second coordinate projection. We have

$$K_3(x, y) = \frac{1}{16\pi^2} \begin{bmatrix} K_3^{11}(x, y) & -K_3^{12}(x, y) \\ -K_3^{21}(x, y) & K_3^{22}(x, y) \end{bmatrix},$$

where

$$\begin{aligned}
K_3^{11}(x, y) &= \int_{-\infty}^{\infty} \int_{\mathbb{R}^6} \frac{e^{itz^2} z \chi(z) e^{iz(|x-u_1|+|y-u_2|)}}{|x-u_1||y-u_2|} (I_1 v_1 E(z) v_2 I_1^T)(u_1, u_2) du_1 du_2 dz \\
K_3^{12}(x, y) &= \int_{-\infty}^{\infty} \int_{\mathbb{R}^6} \frac{e^{itz^2} z \chi(z) e^{iz|x-u_1|-\sqrt{2\mu+z^2}|y-u_2|}}{|x-u_1||y-u_2|} (I_1 v_1 E(z) v_2 I_2^T)(u_1, u_2) du_1 du_2 dz \\
K_3^{21}(x, y) &= \int_{-\infty}^{\infty} \int_{\mathbb{R}^6} \frac{e^{itz^2} z \chi(z) e^{iz|y-u_2|-\sqrt{2\mu+z^2}|x-u_1|}}{|x-u_1||y-u_2|} (I_2 v_1 E(z) v_2 I_1^T)(u_1, u_2) du_1 du_2 dz \\
K_3^{22}(x, y) &= \int_{-\infty}^{\infty} \int_{\mathbb{R}^6} \frac{e^{itz^2} z \chi(z) e^{-\sqrt{2\mu+z^2}(|x-u_1|+|y-u_2|)}}{|x-u_1||y-u_2|} (I_2 v_1 E(z) v_2 I_2^T)(u_1, u_2) du_1 du_2 dz
\end{aligned}$$

The rest of this section is devoted to the proof of

$$(62) \quad \sup_{x, y} |K_3^{ij}(x, y)| \lesssim t^{-3/2}, \quad i, j = 1, 2.$$

First, we consider K_3^{11} . Denote

$$\int_{\mathbb{R}^6} \frac{d}{dz} \left(\chi(z) e^{iz(|x-u_1|+|y-u_2|)} (I_1 v_1 E(z) v_2 I_1^T)(u_1, u_2) \right) \frac{du_1 du_2}{|x-u_1||y-u_2|}$$

by $\mathcal{F}_{x, y}(z)$. By Lemma 14, it suffices to prove that

$$(63) \quad \sup_{x, y} \|\widehat{\mathcal{F}_{x, y}}\|_{L^1} < \infty.$$

Let us concentrate on the term where the derivative hits $\chi(I_1 v_1 E v_2 I_1^T)$ (the term where the derivative hits the exponential is similar):

$$\tilde{\mathcal{F}}_{x, y}(z) = \int_{\mathbb{R}^6} [\chi(z) (I_1 v_1 E(z) v_2 I_1^T)]'(u_1, u_2) \frac{e^{iz(|x-u_1|+|y-u_2|)}}{|x-u_1||y-u_2|} du_1 du_2.$$

Note that

$$\begin{aligned}
\|\widehat{\tilde{\mathcal{F}}_{x, y}}(\xi)\|_{L^1} &= \int_{-\infty}^{\infty} \left| \int_{\mathbb{R}^6} [\chi(I_1 v_1 E v_2 I_1^T)]'(\xi - |x-u_1| - |y-u_2|)(u_1, u_2) \frac{du_1 du_2}{|x-u_1||y-u_2|} \right| d\xi \\
&\leq \int_{\mathbb{R}^6} \int_{-\infty}^{\infty} \left| [\chi(I_1 v_1 E v_2 I_1^T)]'(\xi - |x-u_1| - |y-u_2|)(u_1, u_2) \right| \frac{d\xi du_1 du_2}{|x-u_1||y-u_2|} \\
(64) \quad &= \int_{\mathbb{R}^6} \int_{-\infty}^{\infty} \left| [\chi(I_1 v_1 E v_2 I_1^T)]'(\xi)(u_1, u_2) \right| \frac{d\xi du_1 du_2}{|x-u_1||y-u_2|}
\end{aligned}$$

The second line follows from Minkowski's inequality and Fubini's theorem, the third line follows from a change of variable. Note that $I_1 v_1 E(z) v_2 I_1^T(u_1, u_2)$ is a sum of kernels of the form

$$w_1(u_1) E_{ij}(z)(u_1, u_2) w_2(u_2), \quad i, j = 1, 2, \quad w_1, w_2 \in L^2(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3).$$

Using this and the inequality (for $w \in L^2(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$)

$$\begin{aligned}
(65) \quad \left\| \frac{|w(\cdot)|}{|x-\cdot|} \right\|_2^2 &= \int_{|x-u|<1} \frac{|w(u)|^2}{|x-u|^2} du + \int_{|x-u|>1} \frac{|w(u)|^2}{|x-u|^2} du \\
&\lesssim \int_{|u|<1} \frac{1}{|u|^2} du + \int_{\mathbb{R}^3} |w(u)|^2 du \lesssim 1,
\end{aligned}$$

in (64), we have

$$\|\widehat{\tilde{\mathcal{F}}}_{x,y}(\xi)\|_{L^1} \leq \sum_{i,j=1}^2 \int_{-\infty}^{\infty} \left\| \left(\widehat{\chi E_{ij}} \right)'(\xi) \right\|_{L^2 \rightarrow L^2} d\xi.$$

Therefore, for $\tilde{\mathcal{F}}_{x,y}$, (63) follows from

$$(66) \quad \int_{-\infty}^{\infty} \left\| \left(\widehat{\chi E_{ij}} \right)'(\xi) \right\|_{L^2 \rightarrow L^2} d\xi < \infty, \quad i, j = 1, 2.$$

We shall use the following elementary lemma from [ErdSch].

Lemma 16. *For each $z \in \mathbb{R}$, let $F_1(z)$ and $F_2(z)$ be bounded operators from $L^2(\mathbb{R}^3)$ to $L^2(\mathbb{R}^3)$ with kernels $K_1(z)$ and $K_2(z)$. Suppose that K_1, K_2 both have compact support in z and that $K_j(\cdot)(x, y) \in L^1(\mathbb{R})$ for a.e. $x, y \in \mathbb{R}^3$. Let $F(z) = F_1(z) \circ F_2(z)$ with kernel $K(z)$. Then*

$$\int_{-\infty}^{\infty} \left\| \widehat{K}(\xi) \right\|_{2 \rightarrow 2} d\xi \leq \left[\int_{-\infty}^{\infty} \left\| \widehat{K}_1(\xi) \right\|_{2 \rightarrow 2} d\xi \right] \left[\int_{-\infty}^{\infty} \left\| \widehat{K}_2(\xi) \right\|_{2 \rightarrow 2} d\xi \right].$$

Note that $\frac{d}{dz}[\chi(z)E_{ij}(z)]$ is a sum of operators each of which is a composition of operators from the list below (here χ is a suitably chosen smooth cut-off supported in a small neighborhood of the origin):

$$\begin{aligned} F_1(z) &= \chi(z)(A(z) + S_1)^{-1}, \\ F_2(z) &= \chi(z)E_1(z), \\ F_3(z) &= \chi(z)S_1m(z)^{-1}S_1, \\ F_4(z) &= \chi(z)S_1E_2(z)S_1, \end{aligned}$$

and their z derivatives and appropriate projections. Moreover, we leave it to the reader to check that for each of the combinations that contribute to $E_{ij}(z)$ the hypotheses of Lemma 16 are fulfilled. Therefore, in light of Lemma 16, the following lemma completes the analysis of K_3^{11} .

Lemma 17. *For each of the operators F_j , $j = 1, 2, 3, 4$ above,*

$$(67) \quad \int_{-\infty}^{\infty} \left\| \widehat{F}_j(\xi) \right\|_{2 \rightarrow 2} d\xi < \infty.$$

The same statement is valid for their z derivatives, too.

Proof. We omit the analysis of F_1 and F_3 . Recall that

$$\begin{aligned} F_2(z) &= \chi(z)E_1(z) = \chi(z) \frac{(A(z) + S_1)^{-1} - (A_0 + S_1)^{-1}}{z} \\ &= \chi(z) \sum_{k=1}^{\infty} (-1)^k z^{k-1} (A_0 + S_1)^{-1} [A_1(z)(A_0 + S_1)^{-1}]^k. \end{aligned}$$

Let χ_1 be a smooth cut off function which is equal to 1 in $[-1, 1]$. Note that the support of χ is contained in $[-1, 1]$. We have

$$F_2(z) = \sum_{k=1}^{\infty} (-1)^k \chi(z) z^{k-1} (A_0 + S_1)^{-1} [\chi_1(z)A_1(z)(A_0 + S_1)^{-1}]^k.$$

Using Lemma 16 and Young's inequality, we obtain

$$(68) \quad \int_{-\infty}^{\infty} \left\| \widehat{F_2}(\xi) \right\|_{2 \rightarrow 2} d\xi \leq \sum_{k=1}^{\infty} \|(\chi(\widehat{z})z^{k-1})\|_{L^1} \| (A_0 + S_1)^{-1} \|_{2 \rightarrow 2}^{k+1} \left[\int_{-\infty}^{\infty} \|(\widehat{\chi_1 A_1})(\xi)\|_{2 \rightarrow 2} d\xi \right]^k.$$

By an argument similar to Remark 1 in [ErdSch] it is easy to see that $\|(A_0 + S_1)^{-1}\|$ is bounded on L^2 . Also note that

$$(69) \quad \begin{aligned} \|(\chi(\widehat{z})z^{k-1})\|_{L^1} &\lesssim \|(1 + |\xi|)(\chi(\widehat{z})z^{k-1})(\xi)\|_{L^2} \\ &\lesssim \|\chi(z)z^{k-1}\|_2 + \left\| \frac{d}{dz}(\chi(z)z^{k-1}) \right\|_2 \lesssim \lambda_0^k. \end{aligned}$$

Below, we prove that

$$(70) \quad \int_{-\infty}^{\infty} \|(\widehat{\chi_1 A_1})(\xi)\|_{2 \rightarrow 2} d\xi \lesssim 1.$$

If λ_0 is chosen sufficiently small, using (69) and (70) in (68) completes the proof of the lemma for F_2 .

Recall that

$$\begin{aligned} A_1(z)(x, y) &= \frac{1}{4\pi z|x-y|} v_2(x) \begin{bmatrix} e^{iz|x-y|} - 1 & 0 \\ 0 & -e^{-\sqrt{2\mu+z^2}|x-y|} + e^{-\sqrt{2\mu}|x-y|} \end{bmatrix} v_1(y) \\ &= \frac{1}{4\pi} v_2(x) \begin{bmatrix} i \int_0^1 e^{iz|x-y|b} db & 0 \\ 0 & \int_0^1 \frac{bz}{\sqrt{2\mu+z^2b^2}} e^{-\sqrt{2\mu+z^2b^2}|x-y|} db \end{bmatrix} v_1(y). \end{aligned}$$

We have (with $h_a(z) = \frac{z}{\sqrt{2\mu a^2+z^2}} e^{-\sqrt{2\mu a^2+z^2}}$)

$$\begin{aligned} (\widehat{\chi_1 A_1})(\xi)(x, y) &= \frac{v_2(x)}{4\pi} \begin{bmatrix} i \int_0^1 \widehat{\chi_1}(\xi - |x-y|b) db & 0 \\ 0 & \int_0^1 \int_{-\infty}^{\infty} \widehat{\chi_1}(\xi - b|x-y|\eta) \widehat{h_{|x-y|}}(\eta) d\eta db \end{bmatrix} v_1(y). \end{aligned}$$

Hence by Schur's test, we can bound $\int_{-\infty}^{\infty} \|(\widehat{\chi_1 A_1})(\xi)\|_{2 \rightarrow 2} d\xi$ by a sum of quantities of the form

$$(71) \quad \int_{-\infty}^{\infty} \sup_x \int_{\mathbb{R}^3} \int_0^1 |\widehat{\chi_1}(\xi - |x-y|b)| |w_1(x)| |w_2(y)| db dy d\xi, \text{ and}$$

$$(72) \quad \int_{-\infty}^{\infty} \sup_x \int_{\mathbb{R}^3} \int_0^1 \int_{-\infty}^{\infty} |\widehat{\chi_1}(\xi - \eta|x-y|b)| |\widehat{h_{|x-y|}}(\eta)| |w_1(x)| |w_2(y)| d\eta db dy d\xi,$$

where w_1 and w_2 satisfy

$$(73) \quad |w_1(x)| |w_2(y)| \lesssim \langle x \rangle^{-\beta/2} \langle y \rangle^{-\beta/2} \lesssim \langle x-y \rangle^{-\beta/2}.$$

Using (73) in (71), we obtain

$$\begin{aligned} (71) &\lesssim \int_{-\infty}^{\infty} \sup_x \int_0^1 \int_{\mathbb{R}^3} \langle x-y \rangle^{-\beta/2} |\widehat{\chi_1}(\xi - |x-y|b)| dy db d\xi \\ &= \int_{-\infty}^{\infty} \int_0^1 \int_{\mathbb{R}^3} \langle y \rangle^{-\beta/2} |\widehat{\chi_1}(\xi - |y|b)| dy db d\xi \end{aligned}$$

$$\leq \|\widehat{\chi}_1\|_1 \int_{\mathbb{R}^3} \langle y \rangle^{-\beta/2} dy < \infty$$

provided $\beta > 6$, i.e. $|V(x)| \lesssim \langle x \rangle^{-6-}$. Now, we bound (72). Using (73) (with $\beta = 0$) and (60) in (72), we obtain

$$\begin{aligned} (72) &\lesssim \int_{-\infty}^{\infty} \sup_x \int_{-\infty}^{\infty} \int_0^1 \int_{\mathbb{R}^3} |\widehat{\chi}_1(\xi - \eta|x-y|b)| \frac{e^{-\sqrt{\mu}|x-y|}}{|x-y|\langle \eta \rangle^2} dy db d\eta d\xi \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_0^1 \int_{\mathbb{R}^3} |\widehat{\chi}_1(\xi - \eta|y|b)| \frac{e^{-\sqrt{\mu}|y|}}{|y|\langle \eta \rangle^2} dy db d\eta d\xi \\ &\leq \|\widehat{\chi}_1\|_1 \int_{-\infty}^{\infty} \int_{\mathbb{R}^3} \frac{e^{-\sqrt{\mu}|y|}}{|y|\langle \eta \rangle^2} dy d\eta < \infty. \end{aligned}$$

Next, we consider F_4 :

$$F_4(z) = \chi(z)S_1E_2(z)S_1 = \chi(z)S_1 \sum_{k=1}^{\infty} (-1)^k z^{k-1} m(0)^{-1} [m_1(z)m(0)^{-1}]^k S_1.$$

Arguing as in the case of F_2 , it suffices to prove that

$$(74) \quad \int_{-\infty}^{\infty} \|\widehat{(\chi_1 m_1)}(\xi)\|_{2 \rightarrow 2} d\xi \lesssim 1,$$

where χ_1 is a smooth cut-off function which is equal to 1 in the support of χ (i.e. in $[-\lambda_0, \lambda_0]$) and which is supported in $[-\lambda_1, \lambda_1]$. Recall that

$$m_1(z) = S_1 \frac{A_1(z) - A_1(0)}{z} S_1 + \sum_{j=1}^{\infty} S_1 (-1)^j z^{j-1} (A_1(z)(A_0 + S_1)^{-1})^{j+1} S_1.$$

The second summand can be analyzed as above (here λ_1 is chosen sufficiently small to guarantee the convergence of the series, and then we choose λ_0 even smaller). Now, we consider the first summand. Note that

$$\begin{aligned} (75) \quad A_2(z)(x, y) &:= \frac{A_1(z) - A_1(0)}{z}(x, y) \\ &= \frac{1}{4\pi} v_2(x) \begin{bmatrix} i \int_0^1 \frac{e^{iz|x-y|b} - 1}{z} db & 0 \\ 0 & \int_0^1 \frac{b}{\sqrt{2\mu+z^2b^2}} e^{-\sqrt{2\mu+z^2b^2}|x-y|} db \end{bmatrix} v_1(y) \\ &= \frac{1}{4\pi} v_2(x) \begin{bmatrix} -\int_0^1 |x-y|(1-b)e^{iz|x-y|b} db & 0 \\ 0 & \int_0^1 \frac{b}{\sqrt{2\mu+z^2b^2}} e^{-\sqrt{2\mu+z^2b^2}|x-y|} db \end{bmatrix} v_1(y) \end{aligned}$$

This can be analyzed as in the previous case. Because of the additional $|x-y|$ term, we need to have $\beta > 8$, i.e. $|V(x)| \lesssim \langle x \rangle^{-8-}$.

Next, we deal with $\frac{d}{dz} F_j(z)$. Once again we omit the analysis of F_1 and F_3 . Note that

$$\begin{aligned} \frac{d}{dz} F_2(z) &= \sum_{k=1}^{\infty} (-1)^k \frac{d}{dz} (\chi(z)z^{k-1}) (A_0 + S_1)^{-1} [A_1(z)(A_0 + S_1)^{-1}]^k \\ &\quad + \sum_{k=1}^{\infty} (-1)^k \chi(z)z^{k-1} (A_0 + S_1)^{-1} \times \end{aligned}$$

$$\times \sum_{j=1}^k [A_1(z)(A_0 + S_1)^{-1}]^{j-1} \left[\frac{d}{dz} A_1(z)(A_0 + S_1)^{-1} \right] [A_1(z)(A_0 + S_1)^{-1}]^{k-j}$$

Arguing as above, it suffices to prove that

$$(76) \quad \int_{-\infty}^{\infty} \|(\widehat{\chi_1(A_1)'})'(\xi)\|_{2 \rightarrow 2} d\xi \lesssim 1.$$

Note that (with $a = |x - y|$)

$$\frac{d}{dz} A_1(z)(x, y) = \frac{1}{4\pi} v_2(x) \times \begin{bmatrix} -|x - y| \int_0^1 b e^{iz|x-y|b} db & 0 \\ 0 & \int_0^1 \frac{d}{dz} h_a(abz) db \end{bmatrix} v_1(y)$$

These are similar to the terms treated above. Therefore (76) holds provided $|V(x)| \lesssim \langle x \rangle^{-8-}$.

Finally, we analyze $\frac{d}{dz} F_4(z)$. In view of the preceding, it suffices to prove that

$$(77) \quad \int_{-\infty}^{\infty} \|(\widehat{\chi_1(A_2)'})'(\xi)\|_{2 \rightarrow 2} d\xi \lesssim 1.$$

We have

$$\begin{aligned} \frac{d}{dz} A_2(z)(x, y) = \\ \frac{1}{4\pi} v_2(x) \begin{bmatrix} -i \int_0^1 |x - y|^2 (1 - b) b e^{iz|x-y|b} db & 0 \\ 0 & \int_0^1 \frac{d}{dz} \left[\frac{b}{\sqrt{2\mu + z^2 b^2}} e^{-\sqrt{2\mu + z^2 b^2} |x-y|} \right] db \end{bmatrix} v_1(y) \end{aligned}$$

These are treated as before; (77) holds provided $|V(x)| \lesssim \langle x \rangle^{-10-}$. \square

Now, we consider K_3^{12} . We omit the analysis of the other components of K_3 since they can be handled similarly. Denote

$$\int_{\mathbb{R}^6} \frac{d}{dz} \left(\chi(z) e^{iz|x-u_1| - \sqrt{2\mu+z^2}|y-u_2|} (I_1 v_1 E(z) v_2 I_2^T)(u_1, u_2) \right) \frac{du_1 du_2}{|x - u_1| |y - u_2|}$$

by $\mathcal{G}_{x,y}(z)$. By Lemma 14, it suffices to prove (63) for $\mathcal{G}_{x,y}$. Let us concentrate on the term where the derivative hits $\chi(I_1 v_1 E v_2 I_2^T)$ (the term where the derivative hits the exponential is similar):

$$\tilde{\mathcal{G}}_{x,y}(z) = \int_{\mathbb{R}^6} [\chi(z) (I_1 v_1 E(z) v_2 I_2^T)]'(u_1, u_2) \frac{e^{iz|x-u_1| - \sqrt{2\mu+z^2}|y-u_2|}}{|x - u_1| |y - u_2|} du_1 du_2.$$

Similarly (we denote $e^{-a\sqrt{2\mu+z^2}}$ by e_a)

$$\begin{aligned} \|\widehat{\tilde{\mathcal{G}}_{x,y}}(\xi)\|_{L^1} &\leq \int_{\mathbb{R}^8} \left| [\chi(I_1 v_1 E v_2 I_2^T)]'(\eta)(u_1, u_2) \widehat{e}_{|y-u_2|}(\xi - \eta - |x - u_1|) \right| \frac{d\eta d\xi du_1 du_2}{|x - u_1| |y - u_2|} \\ &\leq \sup_a \|\widehat{e}_a\|_1 \int_{\mathbb{R}^7} \left| [\chi(I_1 v_1 E v_2 I_2^T)]'(\eta)(u_1, u_2) \right| \frac{d\eta du_1 du_2}{|x - u_1| |y - u_2|} \\ &\lesssim \sup_a \|\widehat{e}_a\|_1 \sum_{i,j=1}^2 \int_{-\infty}^{\infty} \left\| (\widehat{\chi E_{ij}})'(\eta) \right\|_{L^2 \rightarrow L^2} d\eta. \end{aligned}$$

It is not difficult to see that (using Lemma 15) $\sup_a \|\widehat{e}_a\|_1 < \infty$. Therefore, for $\tilde{\mathcal{G}}_{x,y}$, (63) follows from (66).

5.2. **The general case.** We now prove Theorem 2 in the general case. Using (47) in (46), we have

$$(78) \quad \begin{aligned} \mathcal{R}(z) &= \mathcal{R}_0(z) - \mathcal{R}_0(z)v_1\Gamma_1(z)v_2\mathcal{R}_0(z) \\ &\quad - \frac{1}{z}\mathcal{R}_0(z)v_1\Gamma_1(z)S_1\Gamma_2(z)S_1\Gamma_1(z)v_2\mathcal{R}_0(z) \\ &\quad - \frac{1}{z^2}(\mathcal{R}_0(z)T(z)\mathcal{R}_0(z) - \mathcal{R}_0(0)T(0)\mathcal{R}_0(0)) \\ &\quad - \frac{1}{z^2}\mathcal{R}_0(0)T(0)\mathcal{R}_0(0), \end{aligned}$$

where $\Gamma_1(z) = (A(z) + S_1)^{-1}$, $\Gamma_2(z) = (m(z) + S_2)^{-1}$ and

$$T(z) = v_1\Gamma_1(z)S_1\Gamma_2(z)S_2b(z)^{-1}S_2\Gamma_2(z)S_1\Gamma_1(z)v_2.$$

Substituting (78) in (49), we have (ignoring $2e^{it\mu}$)

$$(79) \quad K_t = \int_{-\infty}^{\infty} e^{itz^2} \chi(z) z \mathcal{R}_0(z) dz - \int_{-\infty}^{\infty} e^{itz^2} \chi(z) z \mathcal{R}_0(z) v_1 \Gamma_1(z) v_2 \mathcal{R}_0(z) dz$$

$$(80) \quad - \int_{-\infty}^{\infty} e^{itz^2} \chi(z) \mathcal{R}_0(z) v_1 \Gamma_1(z) S_1 \Gamma_2(z) S_1 \Gamma_1(z) v_2 \mathcal{R}_0(z) dz$$

$$(81) \quad - \int_{-\infty}^{\infty} e^{itz^2} \chi(z) \frac{1}{z} (\mathcal{R}_0(z) T(z) \mathcal{R}_0(z) - \mathcal{R}_0(0) T(0) \mathcal{R}_0(0)) dz.$$

Here, the singular term $\frac{1}{z^2} \mathcal{R}_0(0) T(0) \mathcal{R}_0(0)$ in (78) has no contribution since the integral in (49) is a principal value integral and the integrand is odd. The first operator in (79) is dispersive since it is the low energy part of the free evolution. The second operator in (79) is also dispersive, which can be proved by repeating the analysis of K_3 in the previous section. The operator in (80) can be rewritten as a sum of two operators one similar two K_1 and the other similar to K_3 in the previous section. The $L^1 \rightarrow L^\infty$ norm of the former decays like $t^{-1/2}$ and the latter is dispersive. Now, we consider (81). We can write it as a sum of the following operators:

$$(82) \quad \int_{-\infty}^{\infty} e^{itz^2} \chi(z) \mathcal{R}_0(z) \frac{T(z) - T(0)}{z} \mathcal{R}_0(z) dz,$$

$$(83) \quad \int_{-\infty}^{\infty} e^{itz^2} \chi(z) \frac{\mathcal{R}_0(z) - \mathcal{R}_0(0)}{z} T(0) \mathcal{R}_0(z) dz,$$

$$(84) \quad \int_{-\infty}^{\infty} e^{itz^2} \chi(z) \mathcal{R}_0(0) T(0) \frac{\mathcal{R}_0(z) - \mathcal{R}_0(0)}{z} dz.$$

Since we don't have an extra power of z , the $L^1 \rightarrow L^\infty$ norm of these operators decay like $t^{-1/2}$ (see Lemma 14). First let us consider (83). Note that

$$T(0)(x, y) = v_1(x)(S_2b(0)^{-1}S_2)(x, y)v_2(y)$$

is a finite rank operator. Therefore it suffices to study operators with kernel (with the notation $a_1 = |x - u_1|$, $a_2 = |y - u_2|$)

$$\int_{\mathbb{R}^7} \frac{e^{itz^2} \chi(z)}{a_1 a_2} \begin{bmatrix} \frac{e^{iz a_1} - 1}{z} & 0 \\ 0 & \frac{e^{-\sqrt{2\mu} a_1} - e^{-\sqrt{2\mu + z^2} a_1}}{z} \end{bmatrix}$$

$$\begin{bmatrix} r_1(u_1) \\ r_2(u_1) \end{bmatrix} \begin{bmatrix} c_1(u_2) \\ c_2(u_2) \end{bmatrix}^T \begin{bmatrix} e^{iza_2} & 0 \\ 0 & -e^{-\sqrt{2\mu+z^2}a_2} \end{bmatrix} du_1 du_2 dz,$$

where $r_1, r_2, c_1, c_2 \in L^{2, \beta/2}$. This can be rewritten as (with the notation $h_a(z) = \frac{z}{\sqrt{2\mu a^2 + z^2}} e^{-\sqrt{2\mu a^2 + z^2}}$)

$$\int_{\mathbb{R}^7} \int_0^1 \frac{e^{itz^2} \chi(z)}{a_2} \begin{bmatrix} -ir_1(u_1)c_1(u_2)e^{iz(ba_1+a_2)} & ir_1(u_1)c_2(u_2)e^{ibza_1}e^{-\sqrt{2\mu+z^2}a_2} \\ r_2(u_1)c_1(u_2)h_{a_1}(a_1bz)e^{iza_2} & -r_2(u_1)c_2(u_2)h_{a_1}(a_1bz)e^{-\sqrt{2\mu+z^2}a_2} \end{bmatrix} db du_1 du_2 dz.$$

This operator is similar to the operator K_1 studied in the previous section. We omit the analysis. Now, we consider (83). Similarly it suffices to consider operators of the form:

$$\begin{aligned} & \int_{\mathbb{R}^7} \frac{e^{itz^2} \chi(z)}{a_1 a_2} \begin{bmatrix} 1 & 0 \\ 0 & -e^{-\sqrt{2\mu}a_1} \end{bmatrix} \\ & \begin{bmatrix} r_1(u_1) \\ r_2(u_1) \end{bmatrix} \begin{bmatrix} c_1(u_2) \\ c_2(u_2) \end{bmatrix}^T \begin{bmatrix} \frac{e^{iza_2}-1}{z} & 0 \\ 0 & \frac{e^{-\sqrt{2\mu}a_2}-e^{-\sqrt{2\mu+z^2}a_2}}{z} \end{bmatrix} du_1 du_2 dz \\ & = \int_{\mathbb{R}^7} \int_0^1 \frac{e^{itz^2} \chi(z)}{a_1} \begin{bmatrix} -ir_1(u_1)c_1(u_2)e^{ibza_2} & -r_1(u_1)c_2(u_2)h_{a_2}(a_2bz) \\ ir_2(u_1)c_1(u_2)e^{-\sqrt{2\mu}a_1}e^{ibza_2} & r_2(u_1)c_2(u_2)e^{-\sqrt{2\mu}a_1}h_{a_2}(a_2bz) \end{bmatrix} db du_1 du_2 dz. \end{aligned}$$

Once again this operator is similar to K_1 studied in the previous section. Now, we consider (82). We use the following identity

$$\begin{aligned} T(z) - T(0) &= v_1(\Gamma_1(z) - \Gamma_1(0))S_1\Gamma_2(z)S_2b(z)^{-1}S_2\Gamma_2(z)S_1\Gamma_1(z)v_2 \\ &+ v_1S_1(\Gamma_2(z) - \Gamma_2(0))S_2b(z)^{-1}S_2\Gamma_2(z)S_1\Gamma_1(z)v_2 \\ &+ v_1S_2(b(z)^{-1} - b(0)^{-1})S_2\Gamma_2(z)S_1\Gamma_1(z)v_2 \\ &+ v_1S_2b(0)^{-1}S_2(\Gamma_2(z) - \Gamma_2(0))S_1\Gamma_1(z)v_2 \\ &+ v_1S_2b(0)^{-1}S_2(\Gamma_1(z) - \Gamma_1(0))v_2. \end{aligned}$$

In view of the analysis of the operator K_3 in the previous section, it suffices to prove the bound (67) for the following basic building blocks:

$$\begin{aligned} F_1(z) &= \chi(z)\Gamma_1(z) = \chi(z)(A(z) + S_1)^{-1} \\ F_2(z) &= \chi(z)z^{-1}(\Gamma_1(z) - \Gamma_1(0)) = \chi(z)z^{-1}((A(z) + S_1)^{-1} - (A_0 + S_1)^{-1}) \\ F_3(z) &= \chi(z)S_1\Gamma_2(z)S_1 = \chi(z)S_1(m(z) + S_2)^{-1}S_1 \\ F_4(z) &= \chi(z)z^{-1}S_1(\Gamma_2(z) - \Gamma_2(0))S_1 = \chi(z)z^{-1}S_1((m(z) + S_2)^{-1} - (m(0) + S_2)^{-1})S_1 \\ F_5(z) &= \chi(z)S_2b(z)^{-1}S_2 = \chi(z)S_2(b(0) + zb_1(z))^{-1}S_2 \\ F_6(z) &= \chi(z)S_2z^{-1}(b(z)^{-1} - b(0)^{-1})S_2. \end{aligned}$$

The functions F_j with $1 \leq j \leq 4$ were already discussed in Lemma 17. Therefore, it suffices to prove that

$$(85) \quad \max_{j=5,6} \int_{-\infty}^{\infty} \left\| \widehat{F_j}(\xi) \right\|_{2 \rightarrow 2} d\xi < \infty.$$

Recall that, see (18),

$$(86) \quad \begin{aligned} b(0) &= S_2 m_1(0) S_2 \\ b(z) &= b(0) + z b_1(z) = b(0)(1 + z b(0)^{-1} b_1(z)) \\ b_1(z) &= \frac{S_2 [m_1(z) - m_1(0)] S_2}{z} + \frac{1}{z} \sum_{k=1}^{\infty} (-1)^k z^k S_2 (m_1(z)(m(0) + S_2)^{-1})^{k+1} S_2 \end{aligned}$$

$$(87) \quad b(z)^{-1} = \sum_{j=0}^{\infty} (-1)^j z^j (b(0)^{-1} b_1(z))^j b(0)^{-1}.$$

Applying Lemma 16 to the Neuman series in (87) shows that in order to obtain (85), we need to prove that

$$\int_{-\infty}^{\infty} \left\| \chi_1 \widehat{b_1}(\xi) \right\|_{2 \rightarrow 2} d\xi < \infty.$$

Another application of Lemma 16, this time to the Neuman series (86), reduces matters to proving

$$\int_{-\infty}^{\infty} \left\| \chi_2 \widehat{m_1}(\xi) \right\|_{2 \rightarrow 2} d\xi < \infty,$$

which was already done in (74). In both these cases, the cut-off functions χ_1, χ_2 need to be taken with sufficiently small supports. This leaves the term

$$\frac{S_2 [m_1(z) - m_1(0)] S_2}{z}$$

from (86) to be considered. In view of (19) and (75),

$$\begin{aligned} & S_2 \frac{m_1(z) - m_1(0)}{z} S_2 \\ &= S_2 \frac{A_2(z) - A_2(0)}{z} S_2 + \sum_{k=1}^{\infty} (-1)^k z^{k-1} S_2 (A_1(z)(A_0 + S_1)^{-1})^{k+1} S_2. \end{aligned}$$

By (70), and Lemma 16, the Neuman series makes a summable contribution to (85). On the other hand, the contribution of

$$S_2 \frac{A_2(z) - A_2(0)}{z} S_2$$

to (85) is controlled by the bound (77), and we are done.

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