DISPERSIVE ESTIMATES FOR SCHRÖDINGER OPERATORS IN THE PRESENCE OF A RESONANCE AND/OR AN EIGENVALUE AT ZERO ENERGY IN DIMENSION THREE: I

M. BURAK ERDOĞAN AND WILHELM SCHLAG

1. Introduction

Consider the Schrödinger operator $H = -\Delta + V$ in $\mathbb{R}^3$, where V is a real-valued potential. Let P_{ac} be the orthogonal projection onto the absolutely continuous subspace of $L^2(\mathbb{R}^3)$ which is determined by H. In [JouSofSog], [Yaj1], [RodSch], [GolSch] and [Gol], $L^1(\mathbb{R}^3) \to L^\infty(\mathbb{R}^3)$ dispersive estimates for the time evolution $e^{itH}P_{ac}$ were investigated under various decay assumptions on the potential V and the assumption that zero is neither an eigenvalue nor a resonance of H. Recall that zero energy is a resonance iff there is $f \in L^{2,-\sigma}(\mathbb{R}^3) \setminus L^2(\mathbb{R}^3)$ for all $\sigma > \frac{1}{2}$ so that $Hf = 0$. Here $L^{2,-\sigma} = \langle x \rangle^{\sigma} L^2$ are the usual weighted L^2 spaces and $\langle x \rangle := (1 + |x|^2)^{\frac{1}{2}}$.

In the first part of this paper we investigate dispersive estimates when there is a resonance at energy zero. It is well-known, see Rauch [Rau], Jensen and Kato [JenKat], and Murata [Mur], that the decay in that case is $t^{-\frac{1}{2}}$. Moreover, these authors derived expansions of the evolution into inverse powers of time in weighted $L^2(\mathbb{R}^3)$ spaces. Here, we obtain such expansions with respect to the $L^1 \to L^\infty$ norm, albeit only in terms of the powers $t^{-\frac{1}{2}}$ and $t^{-\frac{3}{2}}$. Our results will require decay of the form

$$|V(x)| \leq C\langle x \rangle^{-\beta}, \tag{1}$$

for some $\beta > 0$. Our goal was brevity rather than optimality. In particular, it was not our intention to obtain the minimal value of β, and our results can surely be improved in that regard. Our first result is for the case when zero is only a resonance, but not an eigenvalue.

Date: November 29, 2004.

This work was done in June of 2004, while the first author visited Caltech and he wishes to thank that institution for its hospitality and support. The first author was partially supported by the NSF grant DMS-0303413. The second author was partially supported by a Sloan fellowship and the NSF grant DMS–0300081.

Theorem 1. *Assume that V satisfies (1) with $\beta > 10$ and assume that there is a resonance at energy zero but that zero is not an eigenvalue. Then there is a time dependent rank one operator F_t (see (28) below) such that*

$$\left\| e^{itH} P_{ac} - t^{-1/2} F_t \right\|_{1\to\infty} \le C t^{-3/2},$$

for all $t > 0$ and F_t satisfies

$$\sup_t \|F_t\|_{L^1\to L^\infty} < \infty, \qquad \limsup_{t\to\infty} \|F_t\|_{L^1\to L^\infty} > 0. \tag{2}$$

The following case allows for any combination of resonances and/or eigenvalue at energy zero. It is important to note that the $t^{-\frac{3}{2}}$ bound is destroyed by an eigenvalue at zero, even if zero is not a resonance and even after projecting the zero eigenfunction away (this was discovered by Jensen and Kato [JenKat]).

Theorem 2. *Assume that V satisfies (1) with $\beta > 10$ and assume that there is a resonance at energy zero and/or that zero is an eigenvalue. Then there is a time dependent operator F_t such that*

$$\sup_t \|F_t\|_{L^1\to L^\infty} < \infty, \qquad \left\| e^{itH} P_{ac} - t^{-1/2} F_t \right\|_{1\to\infty} \le C t^{-3/2}.$$

In all cases, the operators F_t can be given explicitly, and they can of course be extracted from our proofs. The methods of this paper also apply to matrix Schrödinger operators, as considered for example in Cuccagna [Cuc] or [Sch]. Details of this will be given elsewhere.

2. Scalar case

Let K_{λ_0} be the operator with kernel

$$K_{\lambda_0}(x,y) = \frac{1}{\pi i} \int_0^\infty e^{it\lambda^2} \lambda \chi_{\lambda_0}(\lambda) [R_V^+(\lambda^2) - R_V^-(\lambda^2)](x,y) d\lambda,$$

where

$$R_V^\pm(\lambda^2) = R_V(\lambda^2 \pm i0) = (H - (\lambda^2 \pm i0))^{-1}$$

is the perturbed resolvent. By the limiting absorption principle, these boundary values are bounded operators on weighted L^2-spaces, see e.g. [Agm]. Here χ is an even smooth function supported in $[-1,1]$ and $\chi(x) = 1$ for $|x| < 1/2$; $\chi_{\lambda_0}(\lambda) = \chi(\lambda/\lambda_0)$. The high energies were studied in [GolSch]:

Theorem 3. [GolSch] *Assume that V satisfies* (1) *with some $\beta > 3$, then for any $\lambda_0 > 0$*

$$\left\| e^{itH} P_{ac} - K_{\lambda_0} \right\|_{1\to\infty} \le C_{\lambda_0} t^{-3/2}.$$

Hence, in the proof of Theorem 1 and Theorem 2, it suffices to consider the operator K_{λ_0} for some λ_0. One can rewrite the kernel of K_{λ_0} as

$$K_{\lambda_0}(x,y) = \frac{1}{\pi i} \int_{-\infty}^{\infty} e^{it\lambda^2} \lambda \chi_{\lambda_0}(\lambda) R_V((\lambda + i0)^2)(x,y) d\lambda, \tag{3}$$

Note that $R((\lambda + i0)^2)(x,y)$ is not an even function of λ; rather, we have

$$R_V((\lambda + i0)^2)(x,y) = \overline{R_V\left((-\lambda + i0)^2\right)(x,y)}.$$

2.1. **Resolvent expansions at zero energy.** In this section, following [JenNen], we obtain resolvent expansions at the threshold $\lambda = 0$ in the presence of a resonance. This is of course similar to Jensen and Kato [JenKat], but we prefer to work with the L^2-based approach from [JenNen]. For $j = 0, 1, 2, ...$, let G_j be the operator with the kernel

$$G_j(x,y) = \frac{1}{4\pi j!} |x - y|^{j-1}.$$

Recall that for each $J = 0, 1, 2, ...$,

$$R_0(\lambda^2) = \sum_{j=0}^{J} (i\lambda)^j G_j + o(\lambda^J), \text{ as } \lambda \to 0. \tag{4}$$

This expansion is valid in the space, $HS_{L^{2,\sigma} \to L^{2,-\sigma}}$, of Hilbert-Schmidt operators between $L^{2,\sigma}$ and $L^{2,-\sigma}$ for $\sigma > \max((2J+1)/2, 3/2)$.

Let $U(x) = 1$ if $V(x) \ge 0$ and $U(x) = -1$ if $V(x) < 0$, $v = |V|^{1/2}$ and $w = vU$. We have

$$V = Uv^2 = wv.$$

We use the symmetric resolvent identity, valid for $\Im\lambda > 0$:

$$R_V(\lambda^2) = R_0(\lambda^2) - R_0(\lambda^2) v A(\lambda)^{-1} v R_0(\lambda^2), \tag{5}$$

where

$$\begin{aligned} A(\lambda) &= U + vR_0(\lambda^2)v = (U + vG_0v) + \lambda \frac{v[R_0(\lambda^2) - G_0]v}{\lambda} \\ &=: A_0 + \lambda A_1(\lambda). \end{aligned} \tag{6}$$

$A_1(\lambda)$ has the kernel

$$A_1(\lambda)(x,y) = \frac{1}{\lambda} v(x) \frac{e^{i\lambda|x-y|}-1}{4\pi|x-y|} v(y),$$
$$|A_1(\lambda)(x,y)| \le \frac{1}{4\pi}|v(x)|\,|v(y)|.$$

Therefore, $A_1(\lambda) \in HS := HS_{L^2\to L^2}$ provided $\langle x\rangle^{\frac{3}{2}+} v(x) \in L^\infty$. Also note that

$$A_1(0) = ivG_1 v = \frac{i\alpha}{4\pi} P_v, \quad \alpha = \|V\|_1,$$

where P_v is the orthogonal projection onto $\mathrm{span}(v)$. It is important to realize that $A(\lambda)$ has a natural meaning for $\lambda \in \mathbb{R}$ via the limit $\mathbb{R} + i0$.

First, we consider the expansions of $A(\lambda)^{-1}$ for λ close to zero as in [JenNen]. The following lemma (Corollary 2.2 in [JenNen]) is our main tool. Note the similarity between (7) and the symmetric resolvent identity.

Lemma 4. [JenNen] *Let $F \subset \mathbb{C}\setminus\{0\}$ have zero as an accumulation point. Let $A(z)$, $z \in F$, be a family of bounded operators of the form*

$$A(z) = A_0 + zA_1(z)$$

with $A_1(z)$ uniformly bounded as $z \to 0$. Suppose that 0 is an isolated point of the spectrum of A_0, and let S be the corresponding Riesz projection. Assume that $\mathrm{rank}(S) < \infty$. *Then for sufficiently small $z \in F$ the operators*

$$B(z) := \frac{1}{z}(S - S(A(z)+S)^{-1}S)$$

are well-defined and bounded on $\mathcal{H}$. Moreover, if $A_0 = A_0^*$, then they are uniformly bounded as $z \to 0$. The operator $A(z)$ has a bounded inverse in $\mathcal{H}$ if and only if $B(z)$ has a bounded inverse in $S\mathcal{H}$, and in this case

$$A(z)^{-1} = (A(z)+S)^{-1} + \frac{1}{z}(A(z)+S)^{-1} S B(z)^{-1} S (A(z)+S)^{-1}. \tag{7}$$

Proof. It is a standard fact that

$$\mathrm{R}an(S) \supset \bigcup_{n=1}^{\infty} \ker(A_0^n).$$

By our assumption $\mathrm{rank}(S) < \infty$ we have equality here, and $(A_0+S)^{-1}$ has a bounded inverse. Hence, $A(z)+S$ also has a bounded inverse for small z, as can be seen from the usual Neuman series. Therefore, $B(z)$ is well-defined for small z and bounded. Moreover, if A_0 is self-adjoint, then

$$S - S(A_0 + S)^{-1}S = 0$$

which implies that $B(z) = O(1)$ as $z \to 0$. Suppose $B(z)$ is invertible on $S\mathcal{H}$. Denote the right-hand side of (7) by $T(z)$. Then

$$\begin{aligned}
T(z)A(z) &= A(z)T(z) \\
&= I + \frac{1}{z}SB(z)^{-1}S(A(z)+S)^{-1} - S(A(z)+S)^{-1} \\
&\quad - \frac{1}{z}S(A(z)+S)^{-1}SB(z)^{-1}S(A(z)+S)^{-1} \\
&= I + \frac{1}{z}SB(z)^{-1}S(A(z)+S)^{-1} - S(A(z)+S)^{-1} \\
&\quad - \frac{1}{z}(S - zB(z))B(z)^{-1}S(A(z)+S)^{-1} = I.
\end{aligned}$$

Conversely, suppose that $A(z)$ is invertible. Define

$$D(z) := z(S + SA(z)^{-1}S) = z(A(z)+S)[A(z)^{-1} - (A(z)+S)^{-1}](A(z)+S).$$

Then

$$\begin{aligned}
B(z)D(z) &= D(z)B(z) \\
&= S + SA(z)^{-1}S - S(A(z)+S)^{-1}S - S(A(z)+S)^{-1}SA(z)^{-1}S \\
&= S + S(A(z)+S)^{-1}SA(z)^{-1}S - S(A(z)+S)^{-1}SA(z)^{-1}S = S,
\end{aligned}$$

so that $D(z)$ is the inverse of $B(z)$ on $S\mathcal{H}$. $\square$

Note that A_0 as in (6) is a compact perturbation of U and that the essential spectrum of U is contained in $\{-1, 1\}$. Moreover, A_0 is self adjoint. Therefore, 0 is an isolated point of the spectrum of A_0 and $\dim(\ker_{A_0}) < \infty$. Let S_1 be the corresponding Riesz projection. Since A_0 is self adjoint, S_1 is the orthogonal projection onto the kernel of A_0 and we have

$$S_1 = (A_0 + S_1)^{-1}S_1 = S_1(A_0 + S_1)^{-1}. \tag{8}$$

Remark 1. By the resolvent identity we have

$$(A_0 + S_1)^{-1} = U - (A_0 + S_1)^{-1}(vG_0v + S_1)U.$$

Since $|V(x)| \lesssim \langle x\rangle^{-3-}$ and S_1 is a finite rank operator, we have $(vG_0v + S_1)U \in HS$, and hence $(A_0 + S_1)^{-1}$ is the sum of U and a Hilbert-Schmidt operator. Therefore, the operator with kernel $|(A_0 + S_1)^{-1}(x, y)|$ is bounded in L^2. This remark will be useful below when we consider dispersive estimates.

We choose $\lambda_0 > 0$ sufficiently small so that $A(\lambda) + S_1$ is invertible for $|\lambda| < \lambda_0$. Using Lemma 4, we see that, for $|\lambda| < \lambda_0$, $A(\lambda)$ is invertible if and only if

$$m(\lambda) = \frac{S_1 - S_1 \left(A(\lambda) + S_1\right)^{-1} S_1}{\lambda}$$

is invertible on S_1L^2 and in this case

$$A(\lambda)^{-1} = (A(\lambda) + S_1)^{-1} + \frac{1}{\lambda}(A(\lambda) + S_1)^{-1}S_1m(\lambda)^{-1}S_1(A(\lambda) + S_1)^{-1}. \tag{9}$$

If λ_0 is sufficiently small, then

$$(A(\lambda) + S_1)^{-1} = (A_0 + S_1)^{-1} + \sum_{k=1}^{\infty}(-1)^k\lambda^k(A_0 + S_1)^{-1}\left(A_1(\lambda)(A_0 + S_1)^{-1}\right)^k.$$

Plugging this into the definition of $m(\lambda)$ and using (8), we obtain

$$\begin{aligned} m(\lambda) &= S_1A_1(\lambda)S_1 + \sum_{k=1}^{\infty}(-1)^k\lambda^kS_1\left(A_1(\lambda)(A_0 + S_1)^{-1}\right)^{k+1}S_1 \\ &= m(0) + \lambda m_1(\lambda), \end{aligned}$$

where

$$m(0) = S_1A_1(0)S_1 = \frac{i\alpha}{4\pi}S_1P_vS_1,$$

$$m_1(\lambda) = S_1\frac{A_1(\lambda) - A_1(0)}{\lambda}S_1 + \sum_{k=1}^{\infty}(-1)^k\lambda^kS_1\left(A_1(\lambda)(A_0 + S_1)^{-1}\right)^{k+1}S_1. \tag{10}$$

If $m(0)$ is invertible in S_1L^2, then we can invert $m(\lambda)$ for small λ using Neuman series and hence obtain an expansion for $A(\lambda)^{-1}$. Since $m(0)$ has rank one, this can only occur if $\mathrm{rank}(S_1) = 1$.

Otherwise, let $S_2 : S_1L^2 \to S_1L^2$ be the orthogonal projection onto the kernel of $m(0)$ where the latter operates on S_1L^2. As above, $m(\lambda) + S_2$ is invertible in S_1L^2 for $|\lambda| < \lambda_0$ (we choose a smaller λ_0 if necessary), and

$$S_2 = S_2(m(0) + S_2)^{-1} = (m(0) + S_2)^{-1}S_2. \tag{11}$$

Lemma 4 asserts that $m(\lambda)$ is invertible on S_1L^2 if and only if

$$b(\lambda) = \frac{S_2 - S_2\,(m(\lambda)+S_2)^{-1}\,S_2}{\lambda}$$

is invertible on S_2L^2 and

$$(12)\ m(\lambda)^{-1} = (m(\lambda)+S_2)^{-1} + \frac{1}{\lambda}\,(m(\lambda)+S_2)^{-1}\,S_2 b(\lambda)^{-1} S_2\,(m(\lambda)+S_2)^{-1}.$$

Note that

$$(m(\lambda)+S_2)^{-1} = (m(0)+S_2)^{-1} + \sum_{k=1}^{\infty}(-1)^k\lambda^k(m(0)+S_2)^{-1}\left(m_1(\lambda)(m(0)+S_2)^{-1}\right)^k.$$

Plugging this into the definition of $b(\lambda)$ and using (11), we obtain

$$\begin{aligned} b(\lambda) &= S_2 m_1(\lambda) S_2 + \sum_{k=1}^{\infty}(-1)^k\lambda^k S_2\left(m_1(\lambda)(m(0)+S_2)^{-1}\right)^{k+1} S_2 \\ &=: b(0) + \lambda b_1(\lambda), \end{aligned}$$

where $b(0) = S_2 m_1(0) S_2$ and

$$(13)\quad b_1(\lambda) = \frac{S_2[m_1(\lambda)-m_1(0)]S_2}{\lambda} + \frac{1}{\lambda}\sum_{k=1}^{\infty}(-1)^k\lambda^k S_2\left(m_1(\lambda)(m(0)+S_2)^{-1}\right)^{k+1} S_2.$$

A simple calculation using (4) (with $J=2$) and $S_2S_1 = S_1S_2 = S_2$ shows that

$$(14)\qquad b(0) = -S_2 v G_2 v S_2.$$

Since $G_2 \in HS_{L^{2,\sigma}\to L^{2,-\sigma}}$ for $\sigma > 5/2$, we have $b(0)\in HS$ if $|V(x)| \lesssim \langle x\rangle^{-5-\varepsilon}$.

Below, we characterize the spaces S_1L^2, S_2L^2 and also prove that $b(0)$ is always invertible on S_2L^2. Therefore, for small λ, $b(\lambda)$ is invertible. This proves that $A(\lambda)$ is invertible for $0<|\lambda|<\lambda_0$. Using (9) and (12), we obtain

$$(15)\qquad \begin{aligned} A(\lambda)^{-1} &= \Gamma_1(\lambda) \\ &\quad + \frac{1}{\lambda}\Gamma_1(\lambda)S_1\Gamma_2(\lambda)S_1\Gamma_1(\lambda) \\ &\quad + \frac{1}{\lambda^2}\Gamma_1(\lambda)S_1\Gamma_2(\lambda)S_2 b(\lambda)^{-1}S_2\Gamma_2(\lambda)S_1\Gamma_1(\lambda), \end{aligned}$$

where

$$\Gamma_1(\lambda) = (A(\lambda)+S_1)^{-1}, \text{ and } \Gamma_2(\lambda) = (m(\lambda)+S_2)^{-1}.$$

Note that this formula is also valid in the case $S_2 = 0$.

Lemma 5. *Assume $|V(x)| \lesssim \langle x\rangle^{-3-\varepsilon}$. Then $f \in S_1L^2\backslash\{0\}$ if and only if $f = wg$ for some $g \in L^{2,-\frac{1}{2}-}\backslash\{0\}$ such that*

$$-\Delta g + Vg = 0 \text{ in } \mathcal{S}'. \tag{16}$$

Proof. First recall that, for $g \in L^{2,-\frac{1}{2}-}$, (16) holds if and only if

$$(I + G_0V)g = 0,$$

see Lemma 2.4 in [JenKat]. Suppose $f \in S_1L^2\backslash\{0\}$. Then

$$(A_0f)(x) = U(x)f(x) + \frac{v(x)}{4\pi}\int \frac{v(y)f(y)}{|x-y|}\,dy = 0$$
$$\Rightarrow\ f(x) + \frac{w(x)}{4\pi}\int \frac{v(y)f(y)}{|x-y|}\,dy = 0. \tag{17}$$

Let

$$g(x) = -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{v(y)f(y)}{|x-y|}\,dy. \tag{18}$$

Note that $g \in L^{2,-\frac{1}{2}-}$ and $f(x) = w(x)g(x)$ for each x. Moreover, (16) holds since

$$g(x) = -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{v(y)f(y)}{|x-y|}\,dy = -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{V(y)g(y)}{|x-y|}\,dy = -[G_0Vg](x).$$

Conversely, assume $f = wg$ for some g as in the hypothesis. Then $f \in L^{2,1+}$ and

$$\begin{aligned} A_0f(x) &= U(x)f(x) + \frac{v(x)}{4\pi}\int \frac{v(y)f(y)}{|x-y|}\,dy \\ &= v(x)g(x) + \frac{v(x)}{4\pi}\int \frac{V(y)g(y)}{|x-y|}\,dy = v(I+G_0V)g = 0. \end{aligned}$$

Note that since g is not identically zero, $Vg \neq 0$, and hence $f \neq 0$. □

By Lemma 5, we see that $f \in S_1L^2$ implies $f \in L^{2,1+}$.

Lemma 6. *Assume $|V(x)| \lesssim \langle x\rangle^{-3-\varepsilon}$. Then $f \in S_2L^2\backslash\{0\}$ if and only if $f = wg$ for some $g \in L^2\backslash\{0\}$ such that*

$$-\Delta g + Vg = 0 \text{ in } \mathcal{S}'.$$

Proof. Suppose $f \in S_2L^2 \backslash \{0\}$. Note that $S_2L^2 \subset S_1L^2$ and, by Lemma 5, we have $f = wg$ for some $g \in L^{2,-\frac{1}{2}-}\backslash\{0\}$ such that $-\Delta g + Vg = 0$ in $\mathcal{S}'$. By the definition of S_2, we have

$$S_1 P_v f = 0.$$

Note that $S_1 P_v f = 0$ if and only if

$$S_1 v = 0 \text{ or } P_v f = 0.$$

In the first case, $S_2 = S_1$ and $P_v f = 0$ for any $f \in S_2L^2$. We have the same conclusion in the second case. Thus,

$$\int_{\mathbb{R}^3} v(y) f(y)\, dy = 0.$$

Using this and (18), we obtain

$$g(x) = -\frac{1}{4\pi} \int_{\mathbb{R}^3} \left[\frac{1}{|x-y|} - \frac{1}{1+|x|}\right] v(y) f(y)\, dy \in L^{2,\frac{1}{2}-}.$$

This is because

$$\left|\frac{1}{|x-y|} - \frac{1}{1+|x|}\right| \le \frac{1+|y|}{|x-y|(1+|x|)} \tag{19}$$

and $f \in L^{2,1+}$.

Conversely, assume $f = wg$ for some g as in the hypothesis. Then

$$g(x) = -\frac{1}{4\pi} \int_{\mathbb{R}^3} \left[\frac{1}{|x-y|} - \frac{1}{1+|x|}\right] v(y) f(y)\, dy - \frac{1}{4\pi(1+|x|)} \int_{\mathbb{R}^3} v(y) f(y)\, dy.$$

By (19) the first summand is in L^2. Therefore

$$\left[\int_{\mathbb{R}^3} v(y) f(y)\, dy\right] \frac{1}{1+|\cdot|} \in L^2(\mathbb{R}^3).$$

Thus, $\int v(y) f(y)\, dy = 0$ and $f \in S_2L^2\backslash\{0\}$. □

Lemma 7. *Assume $|V(x)| \lesssim \langle x\rangle^{-5-\varepsilon}$. Then, as an operator in S_2L^2, the kernel of $b(0)$ is trivial.*

Proof. Assume that for some $f \in S_2L^2$, $b(0)f = 0$, i.e.,

$$\langle G_2 v f, v f\rangle = 0.$$

¿From the proof of Lemma (6), we have

$$\int_{\mathbb{R}^3} f(y)v(y)dy = 0.$$

Using this and (4) (with $J = 2$), we obtain

$$\begin{aligned}
0 &= \langle G_2 vf, vf\rangle \\
&= \lim_{\lambda\to 0} \left\langle \frac{R_0(\lambda^2) - G_0}{\lambda^2} vf, vf \right\rangle \\
&= \lim_{\lambda\to 0} \frac{1}{\lambda^2} \int \left((\xi^2+\lambda^2)^{-1} - \xi^{-2}\right) \widehat{vf}(\xi)\overline{\widehat{vf}}(\xi)d\xi \\
&= \lim_{\lambda\to 0} \int \frac{1}{\xi^2(\xi^2+\lambda^2)} |\widehat{vf}(\xi)|^2 d\xi \\
&= \int \frac{|\widehat{vf}(\xi)|^2}{\xi^4} d\xi \quad \text{(by the Monot. Conv. Thm.)} \\
&= \langle R_0(0)vf, R_0(0)vf\rangle \qquad \Rightarrow \widehat{vf} = 0 \Rightarrow vf = 0.
\end{aligned}$$

Using this in (17), we obtain $f = 0$. □

2.2. **Dispersive estimate when zero is not an eigenvalue.** In this section, we prove Theorem 1. When zero is not an eigenvalue, $S_2 = 0$ and (15) reduces to

$$A(\lambda)^{-1} = (A(\lambda)+S_1)^{-1} + \frac{1}{\lambda}(A(\lambda)+S_1)^{-1}S_1 m(\lambda)^{-1} S_1 (A(\lambda)+S_1)^{-1}, \tag{20}$$

where

$$\begin{aligned}
(A(\lambda)+S_1)^{-1} &= (A_0+S_1)^{-1} + \sum_{k=1}^{\infty} (-1)^k \lambda^k (A_0+S_1)^{-1} \left[A_1(\lambda)(A_0+S_1)^{-1}\right]^k \\
&=: (A_0+S_1)^{-1} + \lambda E_1(\lambda),
\end{aligned}$$

$$\begin{aligned}
m(\lambda)^{-1} &= m(0)^{-1} + \sum_{k=1}^{\infty} (-1)^k \lambda^k m(0)^{-1} \left[m_1(\lambda) m(0)^{-1}\right]^k \\
&=: m(0)^{-1} + \lambda E_2(\lambda).
\end{aligned} \tag{21}$$

Thus, using (8), we obtain

$$\begin{aligned}
A(\lambda)^{-1} = &\frac{1}{\lambda} S_1 m(0)^{-1} S_1 \\
&+ (A(\lambda)+S_1)^{-1} \\
&+ E_1(\lambda) S_1 m(\lambda)^{-1} S_1 \left(A(\lambda)+S_1\right)^{-1}
\end{aligned} \tag{22}$$

$$
\begin{aligned}
&+ \left(A(\lambda)+S_1\right)^{-1} S_1 E_2(\lambda) S_1 \left(A(\lambda)+S_1\right)^{-1} \\
&+ \left(A(\lambda)+S_1\right)^{-1} S_1 m(\lambda)^{-1} S_1 E_1(\lambda) \\
&=: \frac{1}{\lambda} S + E(\lambda).
\end{aligned}
$$

Note that S is a rank one operator. Plugging (22) into (5), we have

$$
\begin{aligned}
R_V(\lambda^2) = &-\frac{1}{\lambda} R_0(\lambda^2) v S v R_0(\lambda^2) \\
&+ R_0(\lambda^2) - R_0(\lambda^2) v E(\lambda) v R_0(\lambda^2).
\end{aligned}
$$

Using this in (3), we get

$$
K_{\lambda_0}(x,y) = K_1(x,y) + K_2(x,y) - K_3(x,y),
$$

where

$$
\begin{aligned}
K_1(x,y) &= \frac{-i}{16\pi^3} \int_{-\infty}^{\infty} \int_{\mathbb{R}^6} e^{it\lambda^2} \chi_{\lambda_0}(\lambda) \frac{e^{i\lambda(|x-u_1|+|y-u_2|)}}{|x-u_1||y-u_2|} v(u_1) S(u_1,u_2) v(u_2) du_1 du_2 d\lambda, \\
K_2(x,y) &= \int_{-\infty}^{\infty} e^{it\lambda^2} \lambda \chi_{\lambda_0}(\lambda) R_0(\lambda^2)(x,y) d\lambda \\
K_3(x,y) &= \int_{-\infty}^{\infty} e^{it\lambda^2} \lambda \chi_{\lambda_0}(\lambda) [R_0(\lambda^2) v E(\lambda) v R_0(\lambda^2)](x,y) d\lambda.
\end{aligned} \tag{23}
$$

First, we deal with K_1. Note that

$$
\begin{aligned}
&K_1(x,y) = \\
&\frac{-i}{16\pi^3} \int_{\mathbb{R}^6} \int_{-\infty}^{\infty} e^{it\lambda^2} \chi_{\lambda_0}(\lambda) \frac{\cos(\lambda(|x-u_1|+|y-u_2|))}{|x-u_1||y-u_2|} v(u_1) S(u_1,u_2) v(u_2) du_1 du_2 d\lambda.
\end{aligned} \tag{24}
$$

We have

$$
\begin{aligned}
\int_{-\infty}^{\infty} t^{1/2} e^{it\lambda^2} \chi_{\lambda_0}(\lambda) \cos(\lambda a) d\lambda &= \int_{-\infty}^{\infty} \left(t^{1/2} e^{it(\cdot)^2}\right)^{\vee} (u) (\chi_{\lambda_0}(\cdot)\cos(\cdot a))^{\wedge}(u) du \\
&= c \int_{-\infty}^{\infty} e^{iu^2/4t} (\widehat{\chi_{\lambda_0}}(u+a) + \widehat{\chi_{\lambda_0}}(u-a)) du
\end{aligned} \tag{25}
$$

$$
\begin{aligned}
&= c\int_{-\infty}^{\infty} e^{i(u^2+a^2)/4t}\cos(\frac{ua}{2t})\widehat{\chi_{\lambda_0}}(u)du\\
&= c\int_{-\infty}^{\infty} e^{i(u^2+a^2)/4t}\widehat{\chi_{\lambda_0}}(u)du\\
&+ c\int_{-\infty}^{\infty} e^{i(u^2+a^2)/4t}(\cos(\frac{ua}{2t})-1)\widehat{\chi_{\lambda_0}}(u)du\\
&=: C_1(t,a)+C_2(t,a).
\end{aligned}
$$

Using this in (24), we obtain

$$
\begin{aligned}
K_1(x,y) &= \frac{-it^{-1/2}}{16\pi^3}\int_{\mathbb{R}^6}\frac{C_1(t,|x-u_1|+|y-u_2|)}{|x-u_1||y-u_2|}v(u_1)S(u_1,u_2)v(u_2)du_1du_2\\
&+ \frac{-it^{-1/2}}{16\pi^3}\int_{\mathbb{R}^6}\frac{C_2(t,|x-u_1|+|y-u_2|)}{|x-u_1||y-u_2|}v(u_1)S(u_1,u_2)v(u_2)du_1du_2\\
&=: K_{11}(x,y)+K_{12}(x,y).
\end{aligned}
$$

Note that

$$|C_2(t,a)|\le c\frac{|a|}{t}.$$

Thus,

$$
\begin{aligned}
(26)\quad |K_{12}(x,y)| &\le ct^{-3/2}\int_{\mathbb{R}^6}\left(\frac{1}{|x-u_1|}+\frac{1}{|y-u_2|}\right)|v(u_1)||v(u_2)||S(u_1,u_2)|du_1du_2\\
&\lesssim t^{-3/2}\left[\left\|\frac{v(\cdot)}{|x-\cdot|}\right\|_2+\left\|\frac{v(\cdot)}{|y-\cdot|}\right\|_2\right]|||S|||_{2\to2}\|v\|_2\\
&\lesssim t^{-3/2}.
\end{aligned}
$$

The last inequality follows from the fact that S is a rank one operator and the following calculation which holds for $v\in L^2\cap L^\infty$;

$$
\begin{aligned}
(27)\quad \left\|\frac{|v(\cdot)|}{|x-\cdot|}\right\|_2^2 &= \int_{|x-u|<1}\frac{|v(u)|^2}{|x-u|^2}du+\int_{|x-u|>1}\frac{|v(u)|^2}{|x-u|^2}du\\
&\lesssim \int_{|u|<1}\frac{1}{|u|^2}du+\int_{\mathbb{R}^3}|v(u)|^2du\lesssim 1.
\end{aligned}
$$

Now, we consider K_{11}. Note that

$$C_1(t,a) = e^{ia^2/4t}h(t),$$

where $h(t)$ is a smooth function which converges to c as t tends to ∞. We have

$$\begin{aligned} K_{11}(x,y) &= \frac{-ih(t)}{16\pi^3 t^{1/2}} \int\limits_{\mathbb{R}^6} \frac{e^{i|x-u_1|^2/4t}e^{i|y-u_2|^2/4t}}{|x-u_1||y-u_2|} v(u_1)S(u_1,u_2)v(u_2)du_1du_2 \\ &\quad - \frac{ih(t)}{16\pi^3 t^{1/2}} \int\limits_{\mathbb{R}^6} \frac{e^{\frac{i(|x-u_1|+|y-u_2|)^2}{4t}} - e^{\frac{i(|x-u_1|^2+|y-u_2|^2)}{4t}}}{|x-u_1||y-u_2|} v(u_1)S(u_1,u_2)v(u_2)du_1du_2 \\ &=: t^{-1/2}F_t(x,y) + K_{112}(x,y). \end{aligned} \tag{28}$$

Since S is a rank one operator, for each t, F_t is a rank one operator. Also note that by a calculation similar to (26), we obtain $\sup_{t,x,y}|F_t(x,y)| \lesssim 1$. Finally, $F_t \neq 0$ for all t, and $\lim_{t\to\infty} F_t$ exists in the weak sense and does not vanish:

$$\lim_{t\to\infty}\langle F_t f, g\rangle = \frac{-ic}{16\pi^3} \int\limits_{\mathbb{R}^{12}} \frac{f(x)\bar{g}(y)}{|x-u_1||y-u_2|} v(u_1)S(u_1,u_2)v(u_2)\, du_1du_2\, dxdy$$

for any $f, g \in \mathcal{S}$. By a similar calculation, the term K_{112} is dispersive since

$$\left| e^{i(|x-u_1|+|y-u_2|)^2/4t} - e^{i(|x-u_1|^2+|y-u_2|^2)/4t} \right| \lesssim \frac{|x-u_1||y-u_2|}{t}.$$

K_2 is the low energy part of the free evolution and hence it is dispersive. The rest of this section is devoted to the proof of

$$\sup_{x,y}|K_3(x,y)| \lesssim t^{-3/2}. \tag{29}$$

Denote

$$\frac{d}{d\lambda}\big(\chi_{\lambda_0}(\lambda)R_0(\lambda^2)vE(\lambda)vR_0(\lambda^2)\big)$$

by $\mathcal{F}_{x,y}(\lambda)$. By integration by parts we obtain

$$K_3(x,y) = \frac{1}{2it}\int\limits_{-\infty}^{\infty} e^{it\lambda^2}\mathcal{F}_{x,y}(\lambda)d\lambda.$$

Using Parseval's formula, we obtain

$$K_3(x,y) = \frac{c}{t^{3/2}}\int\limits_{-\infty}^{\infty} e^{i\xi^2/4t}\widehat{\mathcal{F}_{x,y}}(\xi)d\xi. \tag{30}$$

Thus, it suffices to prove that

$$\sup_{x,y} \|\widehat{\mathcal{F}_{x,y}}\|_{L^1} < \infty. \tag{31}$$

Recall that

$$\mathcal{F}_{x,y}(\lambda) = \int\limits_{\mathbb{R}^6} \frac{d}{d\lambda}\left[\chi_{\lambda_0}(\lambda)E(\lambda)(u_1,u_2)v(u_1)v(u_2)\frac{e^{i\lambda(|x-u_1|+|y-u_2|)}}{|x-u_1||y-u_2|}\right]du_1du_2.$$

Let us concentrate on the term where the derivative hits $\chi_{\lambda_0}(\lambda)E(\lambda)$ (the term where the derivative hits the exponential is similar):

$$\tilde{\mathcal{F}}_{x,y}(\lambda) = \int\limits_{\mathbb{R}^6} [\chi_{\lambda_0}(\lambda)E(\lambda)]'(u_1,u_2)v(u_1)v(u_2)\frac{e^{i\lambda(|x-u_1|+|y-u_2|)}}{|x-u_1||y-u_2|}du_1du_2.$$

Note that

$$\begin{aligned}
\|\widehat{\tilde{\mathcal{F}}_{x,y}}(\xi)\|_{L^1} &= \int\limits_{-\infty}^{\infty}\left|\int\limits_{\mathbb{R}^6}(\widehat{\chi_{\lambda_0}E})'(\xi+|x-u_1|+|y-u_2|)(u_1,u_2)\frac{v(u_1)v(u_2)}{|x-u_1||y-u_2|}du_1du_2\right|d\xi \\
&\le \int\limits_{\mathbb{R}^6}\int\limits_{-\infty}^{\infty}\left|(\widehat{\chi_{\lambda_0}E})'(\xi+|x-u_1|+|y-u_2|)(u_1,u_2)\right|\frac{|v(u_1)||v(u_2)|}{|x-u_1||y-u_2|}d\xi du_1du_2 \\
&= \int\limits_{\mathbb{R}^6}\int\limits_{-\infty}^{\infty}\left|(\widehat{\chi_{\lambda_0}E})'(\xi)(u_1,u_2)\right|\frac{|v(u_1)||v(u_2)|}{|x-u_1||y-u_2|}d\xi du_1du_2 \\
&\le \left\|\frac{|v(\cdot)|}{|x-\cdot|}\right\|_2\left\|\frac{|v(\cdot)|}{|y-\cdot|}\right\|_2\int\limits_{-\infty}^{\infty}\left\|\left|(\widehat{\chi_{\lambda_0}E})'(\xi)\right|\right\|_{L^2\to L^2}d\xi \\
&\lesssim \int\limits_{-\infty}^{\infty}\left\|\left|(\widehat{\chi_{\lambda_0}E})'(\xi)\right|\right\|_{L^2\to L^2}d\xi.
\end{aligned}$$

The second line follows from Minkowski's inequality and Fubini's theorem, the third line follows from a change of variable, and the last line follows from the calculation (27). Therefore, for $\tilde{\mathcal{F}}_{x,y}$, (31) follows from

$$\int\limits_{-\infty}^{\infty}\left\|\left|(\widehat{\chi_{\lambda_0}E})'(\xi)\right|\right\|_{L^2\to L^2}d\xi < \infty. \tag{32}$$

We shall use the following elementary lemma.

Lemma 8. *For each $\lambda \in \mathbb{R}$, let $F_1(\lambda)$ and $F_2(\lambda)$ be bounded operators from $L^2(\mathbb{R}^3)$ to $L^2(\mathbb{R}^3)$ with kernels $K_1(\lambda)$ and $K_2(\lambda)$. Suppose that K_1, K_2 both have compact support in λ and that $K_j(\cdot)(x,y) \in L^1(\mathbb{R})$ for a.e. $x, y \in \mathbb{R}^3$. Let $F(\lambda) = F_1(\lambda) \circ F_2(\lambda)$ with kernel $K(\lambda)$. Then*

$$\int_{-\infty}^{\infty} \Big\| \big|\widehat{K}(\xi)\big| \Big\|_{2\to 2} d\xi \leq \left[\int_{-\infty}^{\infty} \Big\| \big|\widehat{K_1}(\xi)\big| \Big\|_{2\to 2} d\xi\right] \left[\int_{-\infty}^{\infty} \Big\| \big|\widehat{K_2}(\xi)\big| \Big\|_{2\to 2} d\xi\right].$$

Proof. Without loss of generality we can assume that the right hand side is finite. Note that

$$\Big\| \int_{-\infty}^{\infty} |\widehat{K_j}(\xi)| d\xi \Big\|_{2\to 2} \leq \int_{-\infty}^{\infty} \||\widehat{K_j}(\xi)|\|_{2\to 2} d\xi < \infty, \;\; j = 1, 2.$$

This implies that

$$\int_{-\infty}^{\infty} |\widehat{K_j}(\xi)(x,y)| d\xi < \infty, \;\; j = 1, 2,$$

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^2} |\widehat{K}_1(\xi)(x,x_1)||\widehat{K}_2(\eta)(x_1,y)|\, d\xi d\eta dx_1 < \infty \tag{33}$$

$$\sup_{\lambda\in\mathbb{R}} \int_{\mathbb{R}^3} |K_1(\lambda)(x,x_1)K_2(\lambda)(x_1,y)|\, dx_1 < \infty \tag{34}$$

for a.e. $x, y \in \mathbb{R}^3$. By definition, for a.e. $x_1, x_3 \in \mathbb{R}^3$,

$$K(\lambda)(x_1,x_3) = \int_{\mathbb{R}^3} K_1(\lambda)(x_1,x_2)K_2(\lambda)(x_2,x_3)\, dx_2$$

and $K(\cdot)(x_1,x_3) \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$ by (34) and the compact support assumption in λ. Moreover, for a.e. $\xi \in \mathbb{R}$,

$$\hat{K}(\xi)(x_1,x_3) = \int_{\mathbb{R}^3}\int_{-\infty}^{\infty} \hat{K}_1(\xi-\eta)(x_1,x_2)\hat{K}_2(\eta)(x_2,x_3)\, d\eta dx_2. \tag{35}$$

To see this final identity, denote the right-hand side by $F(\xi; x_1, x_3)$. Then $F(\cdot; x_1, x_3) \in L^1(\mathbb{R})$ for a.e. choice of x_1, x_3 by (33), and

$$\int_{-\infty}^{\infty} e^{2\pi i\xi} F(\xi; x_1, x_3)\, d\xi = \int_{-\infty}^{\infty}\int_{\mathbb{R}^3}\int_{-\infty}^{\infty} e^{2\pi i(\xi-\eta)}\hat{K}_1(\xi-\eta)(x_1,x_2)e^{2\pi i\eta}\hat{K}_2(\eta)(x_2,x_3)\, d\eta dx_2 d\xi$$

$$= \int_{\mathbb{R}^3} K_1(\lambda)(x_1, x_2) K_2(\lambda)(x_2, x_3)\, dx_2 d\lambda.$$

The final equality sign here follows by Fubini and since $\hat{K}_j(\cdot)(x, y) \in L^1(\mathbb{R})$ for a.e. choice of x, y by (33). Hence, (35) holds by uniqueness of the Fourier transform. The lemma now follows by putting absolute values inside of (35) and duality. $\square$

Note that $\frac{d}{d\lambda}[\chi_{\lambda_0}(\lambda)E(\lambda)]$ is a sum of operators each of which is a composition of operators from the list below (here $\chi(\lambda)$ is a suitably chosen smooth cutoff supported in $[-\lambda_0, \lambda_0]$):

$$\begin{aligned}
F_1(\lambda) &= \chi(\lambda)(A(\lambda) + S_1)^{-1}, \\
F_2(\lambda) &= \chi(\lambda)E_1(\lambda), \\
F_3(\lambda) &= \chi(\lambda)S_1 m(\lambda)^{-1} S_1, \\
F_4(\lambda) &= \chi(\lambda)S_1 E_2(\lambda) S_1,
\end{aligned}$$

and their λ derivatives. Moreover, we leave it to the reader to check that for each of the combinations that contribute to $E(\lambda)$ the hypotheses of Lemma 8 are fulfilled. Therefore, in light of Lemma 8, the following lemma completes the analysis of K_3.

Lemma 9. *For each of the operators F_j, $j = 1, 2, 3, 4$ above,*

$$\int_{-\infty}^{\infty} \left|\left|\left| \widehat{F_j}(\xi) \right|\right|\right|_{2\to 2} d\xi < \infty.$$

The same statement is valid for their λ derivatives, too.

Proof. We omit the analysis of F_1 and F_3. Recall that

$$\begin{aligned}
F_2(\lambda) = \chi(\lambda)E_1(\lambda) &= \chi(\lambda)\frac{(A(\lambda) + S_1)^{-1} - (A_0 + S_1)^{-1}}{\lambda} \\
&= \chi(\lambda) \sum_{k=1}^{\infty} (-1)^k \lambda^{k-1} (A_0 + S_1)^{-1} \left[A_1(\lambda)(A_0 + S_1)^{-1}\right]^k.
\end{aligned}$$

Let χ_1 be a smooth cut off function which is equal to 1 in $[-1, 1]$. Note that the support of χ is contained in $[-1, 1]$. We have

$$F_2(\lambda) = \sum_{k=1}^{\infty} (-1)^k \chi(\lambda) \lambda^{k-1} (A_0 + S_1)^{-1} \left[\chi_1(\lambda) A_1(\lambda)(A_0 + S_1)^{-1}\right]^k.$$

Using Lemma 8 and Young's inequality, we obtain

$$\int_{-\infty}^{\infty} \left|\left|\left| \widehat{F_2}(\xi) \right|\right|\right|_{2\to 2} d\xi \leq \tag{36}$$

$$\sum_{k=1}^{\infty} \|(\widehat{\chi(\lambda)\lambda^{k-1}})\|_{L^1} |||(A_0+S_1)^{-1}|||_{2\to 2}^{k+1} \left[\int_{-\infty}^{\infty} |||(\widehat{\chi_1 A_1})(\xi)|||_{2\to 2} d\xi \right]^k .$$

By Remark 1, $|(A_0+S_1)^{-1}|$ is bounded on L^2. Also note that

$$\begin{aligned} \|(\widehat{\chi(\lambda)\lambda^{k-1}})\|_{L^1} &\lesssim \|(1+|\xi|)(\widehat{\chi(\lambda)\lambda^{k-1}})(\xi)\|_{L^2} \\ &\lesssim \|\chi(\lambda)\lambda^{k-1}\|_2 + \|\frac{d}{d\lambda}(\chi(\lambda)\lambda^{k-1})\|_2 \\ &\lesssim \lambda_0^k. \end{aligned} \tag{37}$$

Below, we prove that

$$\int_{-\infty}^{\infty} |||(\widehat{\chi_1 A_1})(\xi)|||_{2\to 2} d\xi \lesssim 1. \tag{38}$$

If λ_0 is chosen sufficiently small, using (37) and (38) in (36) completes the proof of the lemma for F_2. Recall that

$$\begin{aligned} A_1(\lambda)(x,y) &= v(x)\frac{e^{i\lambda|x-y|}-1}{4\pi\lambda|x-y|}v(y) \\ &= \frac{1}{4\pi i} v(x)v(y) \int_0^1 e^{i\lambda|x-y|b}\, db. \end{aligned}$$

Therefore,

$$(\widehat{\chi_1 A_1})(\xi)(x,y) = \frac{1}{4\pi i} v(x)v(y) \int_0^1 \widehat{\chi_1}(\xi - |x-y|b)\, db.$$

Hence by Schur's test, we have

$$\int_{-\infty}^{\infty} \left|\left|\left| \widehat{\chi_1 A_1}(\xi) \right|\right|\right|_{2\to 2} d\xi \leq \int_{-\infty}^{\infty} \sup_x \int_{\mathbb{R}^3} \int_0^1 |\widehat{\chi_1}(\xi - |x-y|b)||v(x)||v(y)| db\, dy\, d\xi. \tag{39}$$

Since χ_1 is a Schwarz function, we have (for each $N \in \mathbb{N}$)

$$\int_0^1 |\widehat{\chi_1}(\xi - |x-y|b)|db \lesssim \begin{cases} \langle x-y\rangle^{-1}, & |\xi| < |x-y| \\ \langle \xi\rangle^{-N}, & |\xi| > |x-y| \end{cases} \tag{40}$$

We also have

$$|v(x)||v(y)| \lesssim \langle x\rangle^{-\beta/2}\langle y\rangle^{-\beta/2} \lesssim \langle x-y\rangle^{-\beta/2} \tag{41}$$

Using this inequality and (40) in (39), we obtain

$$\begin{aligned}
(39) &\lesssim \int_{-\infty}^{\infty} \sup_x \int_{\mathbb{R}^3} \langle x-y\rangle^{-\beta/2} \left\{ \begin{array}{ll} \langle x-y\rangle^{-1}, & |\xi| < |x-y| \\ \langle \xi\rangle^{-N}, & |\xi| > |x-y| \end{array} \right\} dy\, d\xi \\
&= \int_{-\infty}^{\infty} \int_{\mathbb{R}^3} \langle y\rangle^{-\beta/2} \left\{ \begin{array}{ll} \langle y\rangle^{-1}, & |\xi| < |y| \\ \langle \xi\rangle^{-N}, & |\xi| > |y| \end{array} \right\} dy\, d\xi \\
&< \infty
\end{aligned}$$

provided $\beta > 6$, i.e. $|V(x)| \lesssim \langle x\rangle^{-6-}$. To see the last inequality, fix ξ and consider the integral in y in the regions $\{|y| < |\xi|\}$ and $\{|y| > |\xi|\}$ separately.

Next, we consider F_4:

$$F_4(\lambda) = \chi(\lambda)S_1E_2(\lambda)S_1 = \chi(\lambda)S_1 \sum_{k=1}^{\infty} (-1)^k \lambda^{k-1} m(0)^{-1} \left[m_1(\lambda)m(0)^{-1}\right]^k S_1.$$

Arguing as in the case of F_2, it suffices to prove that

$$\int_{-\infty}^{\infty} \| |\widehat{(\chi_1 m_1)}(\xi)| \|_{2\to 2} d\xi \lesssim 1, \tag{42}$$

where χ_1 is a smooth cut off function which is equal to 1 in the support of χ (i.e. in $[-\lambda_0, \lambda_0]$) and which is supported in $[-\lambda_1, \lambda_1]$. Recall that

$$m_1(\lambda) = S_1 \frac{A_1(\lambda) - A_1(0)}{\lambda} S_1 + \sum_{j=1}^{\infty} S_1(-1)^j \lambda^{j-1} \left(A_1(\lambda)(A_0 + S_1)^{-1}\right)^{j+1} S_1.$$

The second summand can be analyzed as above (here λ_1 is chosen sufficiently small to guarantee the convergence of the series, and than we choose λ_0 even smaller). Now, we consider the first summand. Note that

$$A_2(\lambda)(x,y) := \frac{A_1(\lambda) - A_1(0)}{\lambda}(x,y) \tag{43}$$

$$= v(x)\frac{e^{i\lambda|x-y|} - i\lambda|x-y| - 1}{\lambda^2|x-y|}v(y)\int_0^1 (1-b)e^{i\lambda|x-y|b}\,db.$$

Therefore, using (40) and (41), we obtain

$$\begin{aligned}\left|\chi_1\widehat{S_1A_2}S_1(\xi)(x,y)\right| &= \left|v(x)|x-y|v(y)\int_0^1 (1-b)\widehat{\chi_1}(\xi - b|x-y|)\,db\right| \\ &\lesssim \langle x-y\rangle^{1-\beta/2}\begin{cases}\langle x-y\rangle^{-1}, & |\xi| < |x-y| \\ \langle \xi\rangle^{-N}, & |\xi| > |x-y|\end{cases}\end{aligned}$$

Using this and Schur's test as before, we have

$$\begin{aligned}\int_{-\infty}^{\infty} \left|\left|\left|\chi_1\widehat{S_1A_2}S_1(\xi)\right|\right|\right|_{2\to 2} d\xi &\lesssim \int_{-\infty}^{\infty} \sup_x \int_{\mathbb{R}^3} \langle x-y\rangle^{1-\beta/2}\left\{\begin{array}{ll}\langle x-y\rangle^{-1}, & |\xi| < |x-y| \\ \langle \xi\rangle^{-N}, & |\xi| > |x-y|\end{array}\right\} dy \\ &< \infty\end{aligned}$$

provided $\beta > 8$, i.e. $|V(x)| \lesssim \langle x\rangle^{-8-}$.

Next, we deal with $\frac{d}{d\lambda}F_j(\lambda)$. Once again we omit the analysis of F_1 and F_3. Note that

$$\begin{aligned}\frac{d}{d\lambda}F_2(\lambda) =& \sum_{k=1}^{\infty}(-1)^k \frac{d}{d\lambda}\left(\chi(\lambda)\lambda^{k-1}\right)(A_0+S_1)^{-1}\left[A_1(\lambda)(A_0+S_1)^{-1}\right]^k \\ &+ \sum_{k=1}^{\infty}(-1)^k\chi(\lambda)\lambda^{k-1}(A_0+S_1)^{-1}\times \\ &\times \sum_{j=1}^{k}[A_1(\lambda)(A_0+S_1)^{-1}]^{j-1}[\frac{d}{d\lambda}A_1(\lambda)(A_0+S_1)^{-1}][A_1(\lambda)(A_0+S_1)^{-1}]^{k-j}\end{aligned}$$

Arguing as above, it suffices to prove that

$$\int_{-\infty}^{\infty} |||(\widehat{\chi_1(A_1)'})(\xi)|||_{2\to 2}d\xi \lesssim 1. \tag{44}$$

Note that

$$\begin{aligned}\frac{d}{d\lambda}A_1(\lambda)(x,y) &= -v(x)\frac{e^{i\lambda|x-y|} - i\lambda|x-y|e^{i\lambda|x-y|} - 1}{\lambda^2|x-y|}\,v(y) \\ &= -v(x)\frac{e^{i\lambda|x-y|} - i\lambda|x-y| - 1}{\lambda^2|x-y|}\,v(y) + iv(x)\frac{e^{i\lambda|x-y|}-1}{\lambda}v(y)\end{aligned}$$

$$= -A_2(\lambda) + i\widetilde{A}_1(\lambda)$$

These are similar to the terms treated above. Therefore (44) holds provided $|V(x)| \lesssim \langle x\rangle^{-8-}$.

Finally, we analyze $\frac{d}{d\lambda}F_4(\lambda)$. In view of the preceding, it suffices to prove that

$$\int_{-\infty}^{\infty} \||(\widehat{\chi_1(A_2)}')(\xi)|\|_{2\to 2} d\xi \lesssim 1. \tag{45}$$

We have

$$\begin{aligned}
\frac{d}{d\lambda}A_2(\lambda)(x,y) =& v(x)i\frac{e^{i\lambda|x-y|}-1}{\lambda^2}v(y) - 2v(x)\frac{e^{i\lambda|x-y|}-i\lambda|x-y|-1}{\lambda^3|x-y|}v(y) \\
=& -2v(x)\frac{e^{i\lambda|x-y|}+\frac{1}{2}\lambda^2|x-y|^2-i\lambda|x-y|-1}{\lambda^3|x-y|}v(y) \\
&+ iv(x)\frac{-i\lambda|x-y|+e^{i\lambda|x-y|}-1}{\lambda^2}v(y)
\end{aligned}$$

These are treated as before; (45) holds provided $|V(x)| \lesssim \langle x\rangle^{-10-}$. □

2.3. **The general case.** We now turn to the proof of Theorem 2. In view of (5), (14), and (15), the coefficient of the λ^{-2} power in (5) equals

$$R_0(0)v\Gamma_1(0)S_1\Gamma_2(0)S_2 b(0)^{-1}S_2\Gamma_2(0)S_1\Gamma_1(0)vR_0(0) = -G_0vS_2[S_2vG_2vS_2]^{-1}S_2vG_0.$$

Lemma 10. *The operator $G_0vS_2[S_2vG_2vS_2]^{-1}S_2vG_0$ equals the orthogonal pojection in $L^2(\mathbb{R}^3)$ onto the eigenspace of $H=-\Delta+V$ at zero energy.*

Proof. Let $\{\psi_j\}_{j=1}^J$ be an orthonormal basis in $\mathrm{Ran}(S_2)$. By Lemmas 5 and 6,

$$\psi_j + wG_0v\psi_j = 0 \quad \forall\, 1\le j\le J$$

and we can write $\psi_j = w\phi_j$ for $1\le j\le J$ where $\phi_2\in L^2$, and

$$\int V\phi_j\,dx = \int v\psi_j\,dx = 0.$$

Moreover, the $\{\phi_j\}_{j=1}^J$ are linearly independent and they satisfy

$$\phi_j + G_0V\phi_j = 0$$

for all $1\le j\le J$. Since for any $f\in L^2(\mathbb{R}^3)$, $S_2f = \sum_{j=1}^J\langle f,\psi_j\rangle\psi_j$, we conclude that

$$S_2vG_0f = \sum_{j=1}^J\langle f, G_0v\psi_j\rangle\psi_j = -\sum_{j=1}^J\langle f,\phi_j\rangle\psi_j.$$

Let $A = \{A_{ij}\}_{i,j=1}^J$ denote the matrix of the Hermitian operator

$$S_2 v G_2 v S_2 = \frac{1}{8\pi} S_2 v(x)|x-y|v(y) S_2$$

relative to the basis $\{\psi_j\}_{j=1}^J$. Since $\int_{\mathbb{R}^3} v\psi_j \, dx = 0$, the proof of Lemma 7 shows that

$$\begin{aligned} A_{ij} &= \langle \psi_i, S_2 v G_2 v S_2 \psi_j \rangle = \langle G_0 v \psi_i, G_0 v \psi_j \rangle \\ &= \langle G_0 V \phi_i, G_0 V \phi_j \rangle = \langle \phi_i, \phi_j \rangle. \end{aligned}$$

Let

$$Q := G_0 v S_2 [S_2 v G_2 v S_2]^{-1} S_2 v G_0.$$

Then for any $f \in L^2(\mathbb{R}^3)$,

$$\begin{aligned} Qf &= -\sum_{j=1}^J G_0 v S_2 [S_2 v G_2 v S_2]^{-1} \psi_j \langle f, \phi_j \rangle \\ &= -\sum_{i,j=1}^J G_0 v S_2 \psi_i (A^{-1})_{ij} \langle f, \phi_j \rangle = \sum_{i,j=1}^J \phi_i (A^{-1})_{ij} \langle f, \phi_j \rangle. \end{aligned}$$

In particular,

$$Q\phi_k = \sum_{i,j=1}^J \phi_i (A^{-1})_{ij} \langle \phi_k, \phi_j \rangle = \sum_{i,j=1}^J \phi_i (A^{-1})_{ij} A_{jk} = \phi_k$$

for all $1 \le k \le J$. The conclusion is that $\operatorname{Ran} Q = \operatorname{span}\{\phi_j\}_{j=1}^J$, and that $Q = \mathrm{Id}$ on $\operatorname{Ran} Q$. Since Q is Hermitian, it is the orthogonal projection onto $\operatorname{span}\{\phi_j\}_{j=1}^J$, as claimed. □

This has the following simple and standard consequence for the spectral measure.

Corollary 11. *Let $-\infty < \lambda_N < \lambda_{N-1} < \ldots < \lambda_1 < \lambda_0 \le 0$ be the finitely many eigenvalues of $H = -\Delta + V$. Let P_{λ_j} denote the orthogonal projection in $L^2(\mathbb{R}^3)$ onto the eigenspace of H corresponding to the eigenvalue λ_j. Then*

$$e^{itH} = \sum_{j=0}^N e^{it\lambda_j} P_{\lambda_j} + \frac{1}{2\pi i} \int_0^\infty e^{it\lambda} [R_V^+(\lambda) - R_V^-(\lambda)] \, d\lambda. \tag{46}$$

Moreover,

$$R_V^+(\lambda) - R_V^-(\lambda) = O_{L^2}(\lambda^{-\frac{1}{2}}) \tag{47}$$

as $\lambda \to 0+$ so that the integral in (46) is absolutely convergent at $\lambda = 0$.

Proof. Start from the expression

$$e^{itH} = \frac{1}{2\pi i}\int_0^\infty e^{it\lambda}[R_V(\lambda+i\epsilon) - R_V(\lambda - i\epsilon)]\,d\lambda,$$

which is valid for all $\epsilon > 0$ (via the spectral theorem, for example). The formula (46) follows by passing to the limit $\epsilon \to 0$. Indeed, the projections arise as Cauchy integrals

$$P_{\lambda_j}\frac{1}{2\pi i}\oint_{\gamma_j}\frac{dz}{z-\lambda_j} = P_{\lambda_j}$$

where γ_j is a small circle surrounding λ_j. We need to invoke Lemma 10 in case $\lambda_0 = 0$, since it determines the coefficient of the z^{-1} singularity in the asymptotic expansion of the resolvent. Once we subtract that singularity, what remains is $O(|z|^{-\frac{1}{2}})$, as claimed. $\square$

The point of Lemma 10 and Corollary 11 is really to prove (47), since (46) is of course obvious. One can also deduce Lemma 10 from the proof of the corollary starting from (46), since the most singular power z^{-1} must lead to the projection onto the eigenspace. However, we have chosen to give these direct proofs.

Proof of Theorem 2. In view of (15),

$$\begin{aligned}A(\lambda)^{-1} &= \Gamma_1(\lambda)\\ &+ \frac{1}{\lambda}\Gamma_1(\lambda)S_1\Gamma_2(\lambda)S_1\Gamma_1(\lambda)\\ &+ \frac{1}{\lambda^2}[\Gamma_1(\lambda)S_1\Gamma_2(\lambda)S_2b(\lambda)^{-1}S_2\Gamma_2(\lambda)S_1\Gamma_1(\lambda) - S_2b(0)^{-1}S_2]\\ &+ \frac{1}{\lambda^2}S_2b(0)^{-1}S_2.\end{aligned}$$

Inserting this into (5) leads to

$$\begin{aligned}R_V(\lambda^2) &= R_0(\lambda^2) - R_0(\lambda^2)v\Gamma_1(\lambda)vR_0(\lambda^2)\\ &- \frac{1}{\lambda}R_0(\lambda^2)v\Gamma_1(\lambda)S_1\Gamma_2(\lambda)S_1\Gamma_1(\lambda)vR_0(\lambda^2) \qquad (48)\\ &- \frac{1}{\lambda^2}R_0(\lambda^2)v[\Gamma_1(\lambda)S_1\Gamma_2(\lambda)S_2b(\lambda)^{-1}S_2\Gamma_2(\lambda)S_1\Gamma_1(\lambda) - S_2b(0)^{-1}S_2]vR_0(\lambda^2) \qquad (49)\\ &- \frac{1}{\lambda^2}(R_0(\lambda^2)-G_0)vS_2b(0)^{-1}S_2vR_0(\lambda^2) - \frac{1}{\lambda^2}G_0vS_2b(0)^{-1}S_2v(R_0(\lambda^2)-G_0) \qquad (50)\\ &- \frac{1}{\lambda^2}P_0. \qquad (51)\end{aligned}$$

The three terms up to and including (48) have already been covered in Subsection 2.2. Indeed, the only difference here is that we need to incorporate S_2 into the expression (21):

$$\begin{aligned} m(\lambda)^{-1} &= (m(0)+S_2)^{-1} + \sum_{k=1}^{\infty}(-1)^k\lambda^k(m(0)+S_2)^{-1}\left[m_1(\lambda)(m(0)+S_2)^{-1}\right]^k \\ &=: (m(0)+S_2)^{-1} + \lambda E_2(\lambda). \end{aligned}$$

The term (51) has been dealt with in Corollary 11. Now, we consider (50). Note that when we plug R_V into (3), then the term corresponding to the first summand in (50) is (with the notation $S = S_2 b(0)^{-1} S_2$, $a_1 = |y-y_1|$ and $a_2 = |x-x_1|+|y-y_1|$)

$$\begin{aligned} &\frac{-1}{\pi i}\int_{-\infty}^{\infty}\int_{\mathbb{R}^6} e^{it\lambda^2}\chi_{\lambda_0}(\lambda)\frac{e^{i\lambda|x-x_1|}-1}{\lambda 4\pi|x-x_1|}\frac{e^{i\lambda|y-y_1|}}{4\pi|y-y_1|}v(x_1)S(x_1,y_1)v(y_1)dx_1dy_1d\lambda \\ &= \frac{-1}{16\pi^3}\int_{-\infty}^{\infty}\int_{\mathbb{R}^6} e^{it\lambda^2}\chi_{\lambda_0}(\lambda)\frac{\sin(\lambda a_2)-\sin(\lambda a_1)}{\lambda|x-x_1||y-y_1|}v(x_1)S(x_1,y_1)v(y_1)dx_1dy_1d\lambda \\ &= \frac{-1}{16\pi^3}\int_{-\infty}^{\infty}\int_{\mathbb{R}^6} e^{it\lambda^2}\chi_{\lambda_0}(\lambda)\int_{a_1}^{a_2}\cos(\lambda b)db\frac{v(x_1)S(x_1,y_1)v(y_1)}{|x-x_1||y-y_1|}dx_1dy_1d\lambda \\ &=: t^{-1/2}F_{1,t}(x,y). \end{aligned}$$

Arguing as in (25), we obtain

$$\begin{aligned} |F_{1,t}(x,y)| &= c\left|\int_{-\infty}^{\infty}\int_{\mathbb{R}^6} e^{iu^2/4t}\int_{a_1}^{a_2}[\widehat{\chi_{\lambda_0}}(u+b)+\widehat{\chi_{\lambda_0}}(u-b)]db\frac{v(x_1)S(x_1,y_1)v(y_1)}{|x-x_1||y-y_1|}dx_1dy_1du\right| \\ &\lesssim \int_{\mathbb{R}^6}\int_{a_1}^{a_2}\int_{-\infty}^{\infty}|\widehat{\chi_{\lambda_0}}(u+b)+\widehat{\chi_{\lambda_0}}(u-b)|\frac{|v(x_1)||S(x_1,y_1)||v(y_1)|}{|x-x_1||y-y_1|}du\,db\,dx_1dy_1 \\ &\lesssim \|\widehat{\chi_{\lambda_0}}\|_1\int_{\mathbb{R}^6}\frac{|v(x_1)||S(x_1,y_1)||v(y_1)|}{|y-y_1|}dx_1dy_1 \\ &\lesssim 1 \end{aligned}$$

This inequality holds independently of t, x and y. Therefore,

$$\sup_t \|F_{1,t}\|_{L^1\to L^\infty} \lesssim 1 \quad \text{and} \quad \lim_{t\to\infty} F_{1,t}(x,y) = c\int_{\mathbb{R}^6}\frac{v(x_1)S(x_1,y_1)v(y_1)}{|y-y_1|}dx_1dy_1.$$

The second summand in (50) can be treated similarly.

Now, we consider (49); it can be written as

$$(49) = \lambda^{-1} R_0(\lambda^2) v E_3(\lambda) v R_0(\lambda^2)$$

with

$$\lambda E_3(\lambda) := -\Gamma_1(\lambda) S_1 \Gamma_2(\lambda) S_2 b(\lambda)^{-1} S_2 \Gamma_2(\lambda) S_1 \Gamma_1(\lambda) + S_2 b(0)^{-1} S_2.$$

Clearly, the terms resulting from E_3 resemble K_3 from (23). However, we do not have an extra λ at our disposal, which implies that instead of (29) we will only obtain a $t^{-\frac{1}{2}}$ power. The details are as follows: if we plug R_V into (3), then the term corresponding to (49) is (up to constants)

$$\int_{-\infty}^{\infty} \int_{\mathbb{R}^6} e^{it\lambda^2} \chi_{\lambda_0}(\lambda) \frac{e^{i\lambda|x-x_1|}}{|x-x_1|} \frac{e^{i\lambda|y-y_1|}}{|y-y_1|} v(x_1) E_3(\lambda)(x_1, y_1) v(y_1)\, dx_1 dy_1 d\lambda.$$

By the arguments that lead from (30) to (32), we conclude that the absolute value of this expression does not exceed

$$|t|^{-\frac{1}{2}} \int_{-\infty}^{\infty} \Big\| \Big| \widehat{\chi_{\lambda_0} E_3}(\xi) \Big| \Big\|_{L^2 \to L^2} d\xi.$$

uniformly in $x, y \in \mathbb{R}^3$. To bound this integral, we use Lemma 8. Write

$$\begin{aligned} E_3(\lambda) &= -\lambda^{-1}(\Gamma_1(\lambda) - \Gamma_1(0)) S_1 \Gamma_2(\lambda) S_2 b(\lambda)^{-1} S_2 \Gamma_2(\lambda) S_1 \Gamma_1(\lambda) \\ &\quad - S_1 \lambda^{-1}(\Gamma_2(\lambda) - \Gamma_2(0)) S_2 b(\lambda)^{-1} S_2 \Gamma_2(\lambda) S_1 \Gamma_1(\lambda) \\ &\quad - S_2 \lambda^{-1}(b(\lambda)^{-1} - b(0)^{-1}) S_2 \Gamma_2(\lambda) S_1 \Gamma_1(\lambda) \\ &\quad - S_2 b(0)^{-1} S_2 \lambda^{-1}(\Gamma_2(\lambda) - \Gamma_2(0)) S_1 \Gamma_1(\lambda) \\ &\quad - S_2 b(0)^{-1} S_2 \lambda^{-1}(\Gamma_1(\lambda) - \Gamma_1(0)). \end{aligned}$$

Consequently, we need to prove the bound of Lemma 9 for the following basic building blocks (we dropped the subscript λ_0):

$F_1(\lambda) = \chi(\lambda)\Gamma_1(\lambda) = \chi(\lambda)(A(\lambda) + S_1)^{-1}$

$F_2(\lambda) = \chi(\lambda)\lambda^{-1}(\Gamma_1(\lambda) - \Gamma_1(0)) = \chi(\lambda)\lambda^{-1}((A(\lambda) + S_1)^{-1} - (A_0 + S_1)^{-1})$

$F_3(\lambda) = \chi(\lambda) S_1 \Gamma_2(\lambda) S_1 = \chi(\lambda) S_1 (m(\lambda) + S_2)^{-1} S_1$

$F_4(\lambda) = \chi(\lambda)\lambda^{-1} S_1(\Gamma_2(\lambda) - \Gamma_2(0)) S_1 = \chi(\lambda)\lambda^{-1} S_1((m(\lambda) + S_2)^{-1} - (m(0) + S_2)^{-1}) S_1$

as well as

$$F_5(\lambda) = \chi(\lambda) S_2 b(\lambda)^{-1} S_2 = \chi(\lambda) S_2 (b(0) + \lambda b_1(\lambda))^{-1} S_2$$

$$F_6(\lambda) = \chi(\lambda)S_2\lambda^{-1}(b(\lambda)^{-1} - b(0)^{-1})S_2.$$

The functions F_j with $1 \le j \le 4$ were already discussed in Lemma 9. The only difference here is the appearance of S_2 in F_3 and F_4 (for the function E_2 see (21)). But this does not effect the bounds from Lemma 9, which implies that we only need to prove the following claims concerning the new terms F_5 and F_6:

$$\max_{j=5,6} \int_{-\infty}^{\infty} \left\| \left| \widehat{F_j}(\xi) \right| \right\|_{2\to 2} d\xi < \infty. \tag{52}$$

Recall that, see (13),

$$b(0) = -S_2 v G_2 v S_2$$

$$b(\lambda) = b(0) + \lambda b_1(\lambda) = b(0)(1 + \lambda b(0)^{-1} b_1(\lambda))$$

$$b_1(\lambda) = \frac{S_2[m_1(\lambda) - m_1(0)]S_2}{\lambda} + \frac{1}{\lambda}\sum_{k=1}^{\infty}(-1)^k \lambda^k S_2 \left(m_1(\lambda)(m(0)+S_2)^{-1}\right)^{k+1} S_2 \tag{53}$$

$$b(\lambda)^{-1} = \sum_{j=0}^{\infty}(-1)^j \lambda^j (b(0)^{-1} b_1(\lambda))^j b(0)^{-1}. \tag{54}$$

Applying Lemma 8 to the Neuman series in (54) shows that in order to obtain (52), we need to prove that

$$\int_{-\infty}^{\infty} \left\| \left| \widehat{\chi_1 b_1}(\xi) \right| \right\|_{2\to 2} d\xi < \infty.$$

Another application of Lemma 8, this time to the Neuman series (53), reduces matters to proving

$$\int_{-\infty}^{\infty} \| \, |\widehat{\chi_2 m_1}(\xi)| \, \|_{2\to 2} \, d\xi < \infty,$$

which was already done in (42). In both these cases, the cut-off functions χ_1, χ_2 need to be taken with sufficiently small supports. This leaves the term

$$\frac{S_2[m_1(\lambda) - m_1(0)]S_2}{\lambda}$$

from (53) to be considered. In view of (10) and (43),

$$\begin{aligned} &S_2 \frac{m_1(\lambda) - m_1(0)}{\lambda} S_2 \\ &= S_2 \frac{A_2(\lambda) - A_2(0)}{\lambda} S_2 + \sum_{k=1}^{\infty}(-1)^k \lambda^{k-1} S_2 \left(A_1(\lambda)(A_0 + S_1)^{-1}\right)^{k+1} S_2. \end{aligned}$$

By (38), and Lemma 8, the Neuman series makes a summable contribution to (52). On the other hand, the contribution of

$$S_2 \frac{A_2(\lambda) - A_2(0)}{\lambda} S_2$$

to (52) is controlled by the bound (45), and we are done. □

References

[Agm] Agmon, S. *Spectral properties of Schrödinger operators and scattering theory.* Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 2 (1975), no. 2, 151–218.

[Cuc] Cuccagna, S. *Stabilization of solutions to nonlinear Schrödinger equations.* Comm. Pure Appl. Math. 54 (2001), no. 9, 1110–1145.

[Gol] Goldberg, M. *Dispersive bounds for the three-dimensional Schrödinger equation with almost critical potentials.* preprint, 2004.

[GolSch] Goldberg, M., Schlag, W. *Dispersive estimates for Schrödinger operators in dimensions one and three.* Comm. Math. Phys. 251, no. 1 (2004), 157 - 178.

[JenKat] Jensen, A., Kato, T. *Spectral properties of Schrödinger operators and time–decay of the wave functions.* Duke Math. J. 46 (1979), no. 3, 583–611.

[JenNen] Jensen, A., Nenciu, G. *A unified approach to resolvent expansions at thresholds.* Rev. Math. Phys. 13 (2001), no.6, 717-754.

[JouSofSog] Journé, J.-L., Soffer, A., Sogge, C. D. *Decay estimates for Schrödinger operators.* Comm. Pure Appl. Math. 44 (1991), no. 5, 573–604.

[Mur] Murata, M. *Asymptotic expansions in time for solutions of Schrödinger-type equations* J. Funct. Anal. 49 (1) (1982), 10–56.

[Rau] Rauch, J. *Local decay of scattering solutions to Schrödinger's equation.* Comm. Math. Phys. 61 (1978), no. 2, 149–168.

[RodSch] Rodnianski, I., Schlag, W. *Time decay for solutions of Schrödinger equations with rough and time-dependent potentials.* Invent. Math. 155 (2004), no. 3, 451–513.

[Sch] Schlag, W. *Stable manifolds for an orbitally unstable NLS*, preprint 2004.

[Yaj1] Yajima, K. *The* $W^{k,p}$*-continuity of wave operators for Schrödinger operators.* J. Math. Soc. Japan 47 (1995), no. 3, 551–581.

Department of Mathematics, University of Illinois, Urbana, IL 61801, U.S.A.

Department of Mathematics, Caltech, Pasadena, CA 91125, U.S.A.