A NOTE ON THE FOURIER TRANSFORM OF FRACTAL MEASURES

M. BURAK ERDOĞAN

1. Introduction

Let μ be a compactly supported non-negative measure in $\mathbf{R}^d$. For $\alpha \in (0, d)$, the α-dimensional energy of μ is defined via (see, e.g., [2])

$$I_\alpha(\mu) := \iint \frac{\mathrm{d}\mu(x)\mathrm{d}\mu(y)}{|x-y|^\alpha} = c_{\alpha,d} \int \frac{|\widehat{\mu}(\xi)|^2}{|\xi|^{d-\alpha}} \mathrm{d}\xi,$$

where $\widehat{\mu}$ is the Fourier transform of the measure μ:

$$\widehat{\mu}(\xi) = \int e^{-ix.\xi} \mathrm{d}\mu(x).$$

We are interested in the behavior of the Fourier transform of measures with finite energy. It is easy to see that $I_\alpha(\mu) < \infty$ does not imply any pointwise decay of $|\widehat{\mu}(\xi)|$ as $|\xi| \to \infty$. However, in general, averages of $\widehat{\mu}(\xi)$ behave much better.

Let Γ be a smooth submanifold of $\mathbf{R}^d$ and let ν_Γ be a smooth surface measure on Γ. One may ask the following general question: Fix $\alpha \in (0, d)$, and assume that $I_\alpha(\mu) = 1$. For which $\beta > 0$

$$\int_\Gamma |\widehat{\mu}(R\xi)|^2 \mathrm{d}\nu_\Gamma(\xi) \leq C_\beta R^{-\beta}, \tag{1}$$

for all $R > 1$?

The following theorem is a slight generalization of a result in [6]. We include a proof in the appendix for the sake of completeness.

Theorem 1. *Let μ be a non-negative measure supported in the unit ball in $\mathbf{R}^d$ with $I_\alpha(\mu) = 1$. Fix $a, b \in (0, d)$ and let ν be a compactly supported probability measure satisfying*

$$|\widehat{\nu}(\xi)| \lesssim |\xi|^{-a} \text{ and } \nu(B(x,r)) \lesssim r^b, \ \forall x, \xi \in \mathbf{R}^d, \ \forall r > 0.$$

Then

$$\int |\widehat{\mu}(R\xi)|^2 \, d\nu(\xi) \lesssim R^{-\max(\min(\alpha,a), \alpha-d+b)}.$$

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The case $\Gamma = S^{d-1} \subset \mathbf{R}^d$ was investigated by several authors [3], [4], [5], [1], [6] and [9] in connection with the continuum version of the Erdos' distance set problem. In this case, Theorem 1 can be applied with $a = (d-1)/2$ and $b = d-1$ but does not give sharp bounds for all α. Sharp bounds for each α are known only in dimension 2, see [9]. We discuss the result of [9] in more detail below. In higher dimensions, the known results are slightly better than the bounds given by Theorem 1, see [1].

The general case that Γ has non-vanishing Gaussian curvature was investigated in [6].

In [9], Wolff obtained the following bound: Fix $\alpha \in (1,2)$, and assume that $I_\alpha(\mu) = 1$. Then for each $\varepsilon > 0$

$$\int_{S^1} |\widehat{\mu}(R\xi)|^2 \mathrm{d}\nu(\xi) \leq C_\varepsilon R^{-\alpha/2+\varepsilon}, \tag{2}$$

for all $R > 1$. This bound is sharp modulo R^ε, c.f. [4]. Sharp bounds for $\alpha \in (0,1)$ are given by Theorem 1 and were first obtained by Mattila [4].

As pointed out in [4], (2) implies that any compact set $E \subset \mathbf{R}^2$ with Hausdorff dimension $> 4/3$ has a positive measure distance set $\Delta(E) = \{|x-y| : x, y \in E\}$.

By the uncertainty principle and duality, (2) follows from the following theorem (see Lemma 1.5 in [9] and the discussion following it). Let $A_R(1)$ be the annulus $\{x \in \mathbf{R}^2 : R-1 < |x| < R+1\}$.

Theorem 2. *([9]) Let $\alpha \in (1,2)$. Let μ be a probability measure supported in the unit ball in $\mathbf{R}^2$. Assume that*

$$\mu(B(x,r)) \leq C_1 r^\alpha \text{ for all } x \in \mathbf{R}^2 \text{ and } r > 1/R. \tag{3}$$

Let f be supported in $A_R(1)$ with L^2 norm 1, and $G = f^\vee$ be its inverse Fourier transform. Then for all $\varepsilon > 0$ and $R > 1$

$$\left| \int G \, d\mu \right| \leq C_\varepsilon C_1^{1/2} R^{\frac{1}{2} - \frac{\alpha}{4} + \varepsilon}. \tag{4}$$

In the first part of the paper, we give a different proof of Wolff's result and extend it in the following direction.

Theorem 3. *Let $\alpha \in (1,2)$. Let μ be a non-negative measure supported in the unit ball in $\mathbf{R}^2$ and satisfying (3). Let f be supported in $A_R(1)$ with L^2 norm 1, and $G = f^\vee$. Then, for each $q \geq 1$, we have*

$$\|G\|_{L^q(\mu)} \leq C_{s,q} C_1^{1/q} R^s, \ \ \forall s > \max\left(\frac{1}{2} - \frac{\alpha}{4}, \frac{1}{4} + \frac{1-\alpha}{2q}, \frac{1}{2} - \frac{\alpha}{q}\right), \ \forall R > 1. \tag{5}$$

Moreover, if $\mu(\mathbf{R}^2) \leq 1$, then for each $q \in [1,2]$, we have

$$\|G\|_{L^q(\mu)} \leq C_\varepsilon C_1^{1/2} R^{\frac{1}{2} - \frac{\alpha}{4} + \varepsilon}, \ \ \forall \varepsilon > 0, \ \ \forall R > 1. \tag{6}$$

Remark 1. The range of s in (5) is sharp modulo endpoint issues. To see this first let $q \geq 2$. Note that in this case $\max(\frac{1}{2} - \frac{\alpha}{4}, \frac{1}{4} + \frac{1-\alpha}{2q}, \frac{1}{2} - \frac{\alpha}{q}) = \max(\frac{1}{4} + \frac{1-\alpha}{2q}, \frac{1}{2} - \frac{\alpha}{q})$. To prove the necessity of the condition $s \geq \frac{1}{4} + \frac{1-\alpha}{2q}$ let f be an L^2 normalized smooth bump function supported in the rectangle $\{x \in \mathbf{R}^2 : |x_2| < R^{1/2}, |x_1 - R| < 1/2\} \subset A_R(1)$ and such that $|f^\vee| > R^{1/4}/100$ on the rectangle $P = \{x \in \mathbf{R}^2 : |x_1| < 1, |x_2| < R^{-1/2}\}$ and $f^\vee$ has a Schwartz decay away from P. Also let $\mathrm{d}\mu(x) = R^{1-\alpha/2}\chi_P(x)\mathrm{d}x$. Note that μ satisfies (3) with $C_1 \approx 1$. To obtain the second condition let $f = R^{-1/2}\chi_{A_R(1)}$, and choose a measure μ with $\mu(B(0, R^{-1})) \geq R^{-\alpha}$. In the case $q < 2$ we have $\max(\frac{1}{2} - \frac{\alpha}{4}, \frac{1}{4} + \frac{1-\alpha}{2q}, \frac{1}{2} - \frac{\alpha}{q}) = \frac{1}{2} - \frac{\alpha}{4}$. To prove the necessity of this condition we modify the first example above. Fix $T \approx R^{(\alpha-1)/2}$ and let

$$F^\vee(x) = T^{-1/2} \sum_{k=1}^{T} f^\vee(x - \frac{k}{T} e_2).$$

Note that F is supported in $A_R(1)$, $\|F\|_2 \approx 1$, and $|F^\vee| \gtrsim R^{1/4}T^{-1/2}$ on the set $S = \cup_{k=1}^{T}(P + \frac{k}{T}e_2)$ (because of the Schwartz decay of $f^\vee$). Finally let $\mathrm{d}\mu(x) = R^{1-\alpha/2}\chi_S(x)\mathrm{d}x$. Note that μ satisfies (3) with $C_1 \approx 1$.

The range of s in (6) and the dependence on C_1 is also sharp modulo endpoints. To see this take the function f in the first example above and let $\mathrm{d}\mu(x) = R^{1/2}\chi_P(x)\mathrm{d}x$. Note that μ is a probability measure and satisfies (3) with $C_1 \approx R^{(\alpha-1)/2}$.

Remark 2. Note that in the first part of the theorem we don't need any additional assumption on the total mass of μ. The claim (5) for $q \in [1, 2)$ follows from the case $q = 2$, Hölder's inequality and the bound $\mu(\mathbf{R}^2) = \mu(B(0,1)) \leq C_1$ which follows from (3). The second claim follows from the first one in the same way by using the additional assumption $\mu(\mathbf{R}^2) \leq 1$ instead of $\mu(\mathbf{R}^2) \leq C_1$. A similar remark is valid for Theorem 5 below.

Remark 3. One can obtain some partial results in higher dimensions analogous to Theorem 3 and Wolff's result (4) by combining the proof of Theorem 3 with the recent parabolic bilinear restriction estimate of Tao [7]. In particular, one can obtain the following partial result in the distance set problem:

$$E \subset \mathbf{R}^d, \text{ compact and } \dim(E) > \frac{d(d+2)}{2(d+1)} \implies |\Delta(E)| > 0.$$

The conjectured exponent is $d/2$, see [2]. Tao's result comes into play in the inequalities (23)-(25) below. Note that (25) is the well-known $L^2 \times L^2 \to L^2$ bilinear restriction estimate. One can use Hölder's inequality with $p > (d+2)/d$ and p' in (23) instead of Cauchy-Schwarz and then use the $L^2 \times L^2 \to L^p$ bilinear restriction estimate of Tao after a parabolic rescaling to estimate the first integral. In fact, one needs a statement which is more general than the main result in [7], namely one needs a bilinear restriction estimate for elliptic surfaces with implicit constants depending on the surface locally uniformly. One can obtain this statement by going through the proof of Theorem 1.1, see the final remark in [7]. We omit the details.

In the second part of the paper, we consider the problem (1) in the case when Γ is a cone in $\mathbf{R}^3$. Let $\Gamma = \{z = (x,t) \in \mathbf{R}^2 \times \mathbf{R} : |x| = t, t \in [1,2]\}$ and ν be the normalized surface measure on Γ. Let $\beta(\alpha)$ be the supremum of all $\beta \geq 0$ such that the inequality

$$\int_\Gamma |\widehat{\mu}(Ru)|^2 \mathrm{d}\nu(u) \leq C_\sigma R^{-\beta} \tag{7}$$

holds for every non-negative measure μ supported in the unit ball in $\mathbf{R}^3$ with $I_\alpha(\mu) = 1$.

In the appendix, we discuss counterexamples which imply that

$$\beta(\alpha) \leq \begin{cases} \alpha & ,\alpha \in (0,1/2] \\ 1/2 & ,\alpha \in [1/2,1] \\ \alpha/2 & ,\alpha \in [1,2] \\ \alpha - 1 & ,\alpha \in [2,3). \end{cases} \tag{8}$$

As one may expect, these exponents are same as the exponents for S^1 for $\alpha < 2$.

Note that the bound

$$\beta(\alpha) \geq \max(\min(\alpha, 1/2), \alpha - 1) \tag{9}$$

follows from Theorem 1.

The following theorem takes care of the remaining case $\alpha \in (1,2)$.

Theorem 4. *$\beta(\alpha) \geq \frac{\alpha}{2}$ for $\alpha \in [1,2]$.*

Theorem 4 follows from the following theorem as in the case of circles.

Let $\Gamma_R(A)$ be the A neighborhood of $R\Gamma$. Let

$$s_0(\alpha, q) = \begin{cases} \max\left(1 - \frac{\alpha}{4}, \frac{3}{4} - \frac{\alpha-1}{2q}, 1 - \frac{\alpha}{q}\right), & \text{for } \alpha \in [1,2], \\ \max\left(\frac{3-\alpha}{2}, \frac{3}{4} + \frac{3-2\alpha}{2q}, 1 - \frac{\alpha}{q}\right), & \text{for } \alpha \in (2,3). \end{cases} \tag{10}$$

Theorem 5. *Let $\alpha \in (1,3)$. Let μ be a non-negative measure supported in the unit ball. Assume that*

$$\mu(B(x,r)) \leq C_1 r^\alpha \text{ for all } x \in \mathbf{R}^3 \text{ and } r > 1/R. \tag{11}$$

Let f be supported in $\Gamma_R(1)$ with L^2 norm 1, and G be its inverse Fourier transform. Then for each $q \geq 1$, we have

$$\|G\|_{L^q(\mu)} \leq C_{s,q} C_1^{1/q} R^s, \quad \forall s > s_0(\alpha, q), \ \forall R > 1. \tag{12}$$

Moreover, if $\mu(\mathbf{R}^3) \le 1$, then for each $q \in [1,2]$, we have

$$\|G\|_{L^q(\mu)} \le C_s C_1^{1/2} R^s, \quad \forall s > s_0(\alpha, 2), \ \forall R > 1.$$

Using (12), one can obtain the following Strichartz type estimates relative to fractal measures for the wave equation in $2+1$ dimensions.

Corollary 1. *Let $\alpha \in (1,3)$, and let μ be a non-negative measure supported in the unit ball satisfying $\mu(B(x,r)) \le r^\alpha$ for all $r > 0$ and $x \in \mathbf{R}^3$. Let u be a solution of*

$$\Box u = 0, \quad u(\cdot, 0) = f, \quad \frac{du}{dt}(\cdot, 0) = g$$

in $\mathbf{R}^3$. Then

$$\|u\|_{L^q(d\mu)} \lesssim \|f\|_{W^{2,s}} + \|g\|_{W^{2,s-1}} \tag{13}$$

for all $s > s_0(\alpha, q)$. Here $\|f\|_{W^{2,s}} = \|(1-\Delta)^{s/2} f\|_2$.

Remark 4. The inequality (13) is already known for $s > \max(3/4, 1-\alpha/4, 1-\alpha/q)$ (see [10] (p.1283-1287) for a nice discussion about this type of inequalities).

Remark 5. The range of s in Theorem 5 is sharp modulo endpoints. The counterexamples are similar to the ones in Remark 1.

Remark 6. For $\alpha \in [2,3]$, the proof of Theorem 5 is relatively easy. Parseval's theorem implies (12) for $q = 2$ and $s > \frac{3}{2} - \frac{\alpha}{2}$. On the other hand L^2 Fourier restriction theory implies (12) for $q = 6$ and $s > 1 - \frac{\alpha}{6}$. It is also easy to see that (12) holds for any q if $s = 1$. Interpolating these bounds, we obtain (12).

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2. List of notation

χ_A: Characteristic function of the set A.
$B(x,r) := \{y : |x-y| < r\}$.
$A_R(C) := \{x \in \mathbf{R}^2 : R - C < |x| < R + C\}$.
$\Gamma_R(C) := \{(x,t) \in \mathbf{R}^2 \times \mathbf{R} : t \in [R, 2R], ||x| - t| \le C\}$.
If P is a rectangle of dimensions $a_1 \times a_2 \times ... \times a_d$ in $\mathbf{R}^d$, then

CP is the rectangle of dimensions $Ca_1 \times Ca_2 ... \times Ca_d$ with the same center and axis directions as P.

A dual rectangle of P is a rectangle with the same axis directions and dimensions $a_1^{-1} \times a_2^{-1} \times ... \times a_d^{-1}$.

a_P is a fixed affine map from $\mathbf{R}^d$ to $\mathbf{R}^d$ which takes the unit cube Q to P.
φ: A fixed Schwartz function (for each dimension d) which is equal to 1 in Q and vanishes outside $2Q$, moreover $\varphi^\vee$ satisfies for all $M > 0$

$$|\varphi^\vee(\xi)| \le C_M \sum_{j=1}^{\infty} 2^{-Mj} \chi_{2^j Q}(\xi), \quad \forall \xi \in \mathbf{R}^d. \tag{14}$$

ϕ: A fixed non-negative Schwartz function satisfying: i) $\phi > 1/2$ in Q, ii) $\widehat{\phi}$ is supported in Q, iii) the inequality (14).
$\varphi_P := \varphi \circ a_P^{-1}$.
$\phi_P := \phi \circ a_P^{-1}$.
C: A constant which may vary from line to line.
$A \lesssim B$: $A \le CB$.
$A \approx B$: $A \lesssim B$ and $B \lesssim A$.
$|A|$: length of the vector A or the measure of the set A.

3. Proof of Theorem 3

In the proof of the theorem, we make repeated use of the following lemma.

Lemma 3.1. *Let $\alpha \in [1,2]$. Let μ be a non-negative measure in $\mathbf{R}^2$ satisfying (3) with $C_1 = 1$. Let D be a rectangle of dimensions $R_1 \times R_2$ such that $R_1 \le R_2 \lesssim R$. Let D_{dual} be the dual of D centered at the origin. Then the function $\mu_D := |\varphi_D^\vee| * \mu$ satisfy*
I) $\|\mu_D\|_\infty \lesssim R_2^{2-\alpha}$,
II) $\|\mu_D\|_1 \lesssim 1$,
III) $\mu_D(x + KD_{dual}) := \int_{KD_{dual}} \mu_D(x+y)dy \lesssim K^\alpha R_2^{1-\alpha} R_1^{-1}$, $\forall K \gtrsim 1$ and $x \in \mathbf{R}^2$.

Proof. Fix $M > 100$. Using (14), we obtain

$$|\varphi_D^\vee(x)| \lesssim R_1 R_2 \sum_{j=1}^{\infty} 2^{-Mj} \chi_{2^j D_{dual}}(x). \tag{15}$$

I) Using (15), we obtain

$$\begin{aligned} |\varphi_D^\vee| * \mu(x) &\lesssim R_1 R_2 \sum_j 2^{-Mj} \int \chi_{2^j D_{dual}}(x-y) \mathrm{d}\mu(y) \\ &\lesssim R_1 R_2 \sum_j 2^{-Mj} (2^j R_2^{-1})^\alpha \frac{R_2}{R_1} \lesssim R_2^{2-\alpha}. \end{aligned}$$

The second inequality follows from (3) and the observation that $2^j D_{dual}$ can be covered by $\lesssim R_2/R_1$ many balls of radius $2^j R_2^{-1}$.

II) follows from Young's inequality using $\|\varphi_D^\vee\|_1 \lesssim 1$.

III) Without loss of generality assume $x = 0$. Using (15) and (3) like above, we obtain

$$\begin{aligned}
\mu_D(KD_{dual}) &\lesssim R_1 R_2 \sum_j 2^{-Mj} \int\int \chi_{KD_{dual}}(y)\chi_{2^j D_{dual}}(y-u)\mathrm{d}\mu(u)\mathrm{d}y \\
&\lesssim R_1 R_2 \sum_j 2^{-Mj} \int\int \chi_{(K+2^j)D_{dual}}(u)\chi_{2^j D_{dual}}(y-u)\mathrm{d}y\mathrm{d}\mu(u) \\
&\lesssim R_1 R_2 \sum_j 2^{-Mj} \left[(K+2^j)^\alpha R_2^{-\alpha}\frac{R_2}{R_1}\right]\left[\frac{2^{2j}}{R_1 R_2}\right] \\
&\lesssim K^\alpha R_2^{1-\alpha} R_1^{-1}.
\end{aligned}$$

□

We also need the following well-known geometric lemma about the size of intersection of circular annuli. Given interval $J \subset [-1/2, 1/2]$, let $A_R(C, J) = \{(\rho\cos(\theta), \rho\sin(\theta)) \in A_R(C) : \theta \in J\}$.

Lemma 3.2. *Let $J_1, J_2 \subset [-1/2, 1/2]$ be two intervals of length $\ell \gtrsim R^{-1/2}$. Assume that the distance between J_1 and J_2 is $\gtrsim \ell$, then for any $x \in \mathbf{R}^2$*

$$|(x + A_R(1, J_1)) \cap A_R(1, J_2)| \lesssim \ell^{-1}.$$

Proof. In the case $\ell \approx R^{-1/2}$ the statement is void since $|A_R(1, J)| \lesssim \ell R \lesssim R^{1/2} \lesssim \ell^{-1}$. Assume $\ell > 1000R^{-1/2}$, and fix x, J_1, J_2. Note that the set $A_1 \cap A_2 := (x + A_R(1, J_1)) \cap A_R(1, J_2)$ has at most two connected components. Let $\mathcal{C}$ be a connected component. It suffices to prove that $\mathrm{diam}(\mathcal{C}) \lesssim \ell^{-1}$. Take a point $y \in \mathcal{C}$. Take an infinite strip $\mathcal{S}_1$ ($\mathcal{S}_2$, *resp.*) of thickness 10 tangent to A_1 (A_2, *resp.*) at the point y. By the hypothesis the angle between the directions of the strips $\mathcal{S}_1, \mathcal{S}_2$ is $\geq \ell/10 \geq 100R^{-1/2}$. Hence, $\mathrm{diam}(\mathcal{S}_1 \cap \mathcal{S}_2) \leq 10\ell^{-1} \leq R^{1/2}/100$. Also note that $A_i \cap B(y, R^{1/2}) \subset \mathcal{S}_i$ for $i = 1, 2$. Since $\mathcal{C}$ is connected, it follows that $\mathcal{C} \subset \mathcal{S}_1 \cap \mathcal{S}_2$. Thus $\mathrm{diam}(\mathcal{C}) \lesssim \ell^{-1}$ □

Proof of Theorem 3. It suffices to prove the theorem for $q \geq 2$ (see Remark 2 in the introduction). We give a proof for $q = 2$ only. The proof for $q > 2$ can be obtained by modifying the proof for $q = 2$ as in the proof of Theorem 5 below. Note that without loss of generality we can let $C_1 = 1$. Also note that it suffices to give a proof for $A_R(1, [-1/2, 1/2])$ instead of $A_R(1)$.

Let f be a function supported in $A_R(1, [-1/2, 1/2])$ with $\|f\|_2 = 1$. We utilize the bilinear approach (see, e.g., [8], [11]). Consider the set of dyadic intervals in $[-1/2, 1/2]$. We say two dyadic intervals I, J are related, $I \sim J$, if i) they have the same length, ii) they are not adjacent and iii) their parents are adjacent. Note that

$$[-1/2,1/2]\times[-1/2,1/2]=\Bigg[\bigcup_{1\leq 2^n\leq R^{1/2}}\Bigg[\bigcup_{|I|=|J|=2^{-n},I\sim J}(I\times J)\Bigg]\Bigg]\bigcup\mathcal{D}. \tag{16}$$

Here $\mathcal{D}$ is a subset of the $CR^{-1/2}$ neighborhood of the diagonal $\{(x,x):x\in[-1/2,1/2]\}$ which can be written as a union of a set of finitely overlapping boxes $I\times I$ of side length $\approx R^{-1/2}$. Let $f_I:=f\chi_{A_R(1,I)}$. Using the decomposition (16), it is easy to see that

$$(f^\vee)^2(\xi)=\sum_n\sum_{|I|=|J|=2^{-n},I\sim J}f_I^\vee(\xi)f_J^\vee(\xi)+Error, \tag{17}$$

where

$$|Error|\lesssim\sum_{I\in I_E}|f_I^\vee|^2.$$

Here I_E is a set of finitely overlapping intervals of length $\approx R^{-1/2}$. By "finitely overlapping", we mean that $\|\sum_{I\in I_E}\chi_I\|_\infty\lesssim 1$. Using (17), we have

$$\begin{aligned}\|f^\vee\|^2_{L^2(\mu)}\leq&\sum_{n=1}^{\log(R^{1/2})}\sum_{|I|=|J|=2^{-n},I\sim J}\|f_I^\vee f_J^\vee\|_{L^1(\mu)}+\sum_{I\in I_E}\|f_I^\vee\|^2_{L^2(\mu)}\\ =:&\ S_1+S_2.\end{aligned} \tag{18}$$

Note that for each $I\in I_E$, the support of f_I, $A_R(1,I)$, is contained in a rectangle D of dimensions $C\times CR^{1/2}$. Hence $f_I^\vee=f_I^\vee*\varphi_D^\vee$. Using this and Hölder's inequality, we have

$$|f_I^\vee|\leq(|f_I^\vee|^2*|\varphi_D^\vee|)^{1/2}\|\varphi_D^\vee\|_1^{1/2}\lesssim(|f_I^\vee|^2*|\varphi_D^\vee|)^{1/2}.$$

Using this and Fubini's theorem we obtain

$$\|f_I^\vee\|^2_{L^2(\mu)}\leq\int|f_I^\vee(x)|^2(\mu*|\varphi_D^\vee|)(x)dx\lesssim\|f_I\|_2^2R^{1-\alpha/2}. \tag{19}$$

In the last inequality, we used Lemma 3.1(I) and Parseval. Since the intervals in I_E are finitely overlapping, (19) implies that

$$S_2\lesssim R^{1-\alpha/2}.$$

To complete the proof of the theorem, we should obtain the same bound for S_1. Since there are $\lesssim\log(R)$ values of n and orthogonality (see, e.g., [8], [11]), it suffices to prove that for each n and for each pair $I\sim J$, $|I|=|J|=2^{-n}$,

$$\|f_I^\vee f_J^\vee\|_{L^1(\mu)}\lesssim R^{1-\alpha/2}\|f_I\|_2\|f_J\|_2, \tag{20}$$

where the implicit constant is independent of I, J and R. First note that the union of the supports of f_I and f_J are contained in a rectangle of dimensions $CR2^{-n}\times CR2^{-2n}$.

Hence $f_I * f_J$ is supported in a rectangle D of dimensions $2CR2^{-n} \times 2CR2^{-2n}$, the longer axis being in the direction e, say. Using $f_I * f_J = (f_I * f_J)\varphi_D$ like above, we obtain

$$\|f_I^\vee f_J^\vee\|_{L^1(\mu)} \leq \int |f_I^\vee(x) f_J^\vee(x)| (\mu * |\varphi_D^\vee|)(x) dx. \tag{21}$$

Consider a tiling of $\mathbf{R}^2$ with rectangles P of dimensions $100 \times 100\ 2^{-n}$, the short axis being in the direction of e. Note that each P is contained in a rectangle $x_P + CR2^{-2n} D_{dual}$ for some $x_P \in \mathbf{R}^2$. Using the properties of the function ϕ, we obtain

$$1 \lesssim \sum_P \phi_P^3 \lesssim \sum_P \phi_P^2 \lesssim 1. \tag{22}$$

Let $f_{I,P} := \widehat{f_I^\vee \phi_P}$. Using (22) in (21) and then Cauchy-Schwarz, we get

$$\begin{aligned}(21) &\lesssim \sum_P \int |f_{I,P}^\vee(x) f_{J,P}^\vee(x)| (\mu * |\varphi_D^\vee|)(x) \phi_P(x) \mathrm{d}x \\ &\lesssim \sum_P \left[\int |f_{I,P}^\vee(x) f_{J,P}^\vee(x)|^2 \mathrm{d}x\right]^{1/2} \left[\int \left[(\mu * |\varphi_D^\vee|)(x)\phi_P(x)\right]^2 \mathrm{d}x\right]^{1/2}. \qquad (23)\end{aligned}$$

To estimate the first integral in (23), we use a well-known L^4 orthogonality argument. Let $A_{I,P}$ be the support of $f_{I,P}$. By Parseval, Cauchy-Schwarz and Young's inequality

$$\begin{aligned}\int |f_{I,P}^\vee(x) f_{J,P}^\vee(x)|^2 \mathrm{d}x &= \int |f_{I,P} * f_{J,P}(\xi)|^2 \mathrm{d}\xi \\ &\lesssim \| |(\xi + A_{I,P}) \cap A_{J,P}| \|_{L^\infty(\xi)} \int (|f_{I,P}|^2 * |f_{I,P}|^2)(\xi) \mathrm{d}\xi \\ &\lesssim \| |(\xi + A_{I,P}) \cap A_{J,P}| \|_{L^\infty(\xi)} \|f_{I,P}\|_2^2 \, \|f_{J,P}\|_2^2. \qquad (24)\end{aligned}$$

Note that $f_{I,P} = f_I * \widehat{\phi_P}$. Hence $A_{I,P}$ is contained in $\mathrm{supp}(f_I) + \mathrm{supp}(\phi_P) = \mathrm{supp}(f_I) + P_{dual}$, where P_{Dual} is the dual of P centered at the origin. At this point the crucial observation is the following:

$$\mathrm{supp}(f_I) + P_{dual} \subset A_R(10, \frac{11}{10} I).$$

Thus, Lemma 3.2 implies that $\| |(\xi + A_{I,P}) \cap A_{J,P}| \|_{L^\infty(\xi)} \lesssim |I|^{-1} = 2^n$. Using this in (24), we see that

$$\int |f_{I,P}^\vee(x) f_{J,P}^\vee(x)|^2 \mathrm{d}x \lesssim 2^n \|f_{I,P}\|_2^2 \, \|f_{J,P}\|_2^2. \tag{25}$$

Now, we obtain a bound for the second integral in (23). This is just a simple application of Lemma 3.1. First note that by Lemma 3.1(I), we have

$$\|\mu * |\varphi_D^\vee|\|_\infty \lesssim R^{2-\alpha} 2^{n\alpha - 2n}. \tag{26}$$

Second, using (14) for ϕ_P and Lemma 3.1(III) (remember that P is contained in $x_P + CR2^{-2n}D_{dual}$ for some $x_P \in \mathbf{R}^2$), we have

$$\int (\mu * |\varphi_D^\vee|)(x)\phi_P(x)\mathrm{d}x \le \sum_{j=1}^{\infty} 2^{-Mj} \int (\mu * |\varphi_D^\vee|)(x)\chi_{2^jP}(x)\mathrm{d}x$$

$$\lesssim \sum_{j=1}^{\infty} 2^{-Mj} 2^{n-n\alpha} 2^{j\alpha} \lesssim 2^{n-n\alpha}. \tag{27}$$

Using (26) and (27), we have

$$\int \left[(\mu * |\varphi_D^\vee|)(x)\phi_P(x)\right]^2 \mathrm{d}x \lesssim \|\mu * |\varphi_D^\vee|\|_\infty \int (\mu * |\varphi_D^\vee|)(x)\phi_P(x)\mathrm{d}x$$

$$\lesssim R^{2-\alpha}2^{-n}. \tag{28}$$

Substituting (25) and (28) into (23) yields (20):

$$\begin{aligned} \|f_I^\vee f_J^\vee\|_{L^1(\mu)} &\lesssim R^{1-\alpha/2} \sum_P \|f_{I,P}\|_2 \|f_{J,P}\|_2 \\ &\lesssim R^{1-\alpha/2} \left[\sum_P \|f_{I,P}\|_2^2\right]^{1/2} \left[\sum_P \|f_{J,P}\|_2^2\right]^{1/2} \\ &\lesssim R^{1-\alpha/2} \|f_I\|_2 \|f_J\|_2. \end{aligned}$$

In the last inequality, we used Parseval's theorem and (22). □

4. Proof of Theorem 5

We prove the theorem for $\alpha \in [1, 2]$ only (see Remark 6 in the introduction). We have the following analog of Lemma 3.1.

Lemma 4.1. *Let $\alpha \in [1, 2]$. Let μ be a non-negative measure in $\mathbf{R}^3$ satisfying (11) with $C_1 = 1$. Let D be a rectangle of dimensions $R_1 \times R_2 \times R_3$ such that $R_1 \le R_2 \le R_3 \lesssim R$. Let D_{dual} be the dual of D centered at the origin. Then the function $\mu_D := |\varphi_D^\vee| * \mu$ satisfy*

I) $\|\mu_D\|_\infty \lesssim R_2^{2-\alpha} R_3$,
II) $\|\mu_D\|_1 \lesssim 1$,
III) $\mu_D(x + KD_{dual}) := \int_{KD_{dual}} \mu_D(x + y)dy \lesssim K^\alpha R_2^{1-\alpha} R_1^{-1}$, $\forall K \gtrsim 1$ and $x \in \mathbf{R}^3$.

We omit the proof since it is similar to the proof of Lemma 3.1.

Proof of Theorem 5. The proof is similar to the proof of Theorem 3. We can assume that $q \ge 2$ and $C_1 = 1$. Let $\Gamma_R(C, J) := \{(\rho\cos(\theta), \rho\sin(\theta), t) \in \Gamma_R(C) : \theta \in J\}$. It suffices to prove the theorem with $\Gamma_R(1, [-1/2, 1/2])$ instead of $\Gamma_R(1)$. Let f be a function supported in $\Gamma_R(1,[-1/2,1/2])$ with $\|f\|_2 = 1$. Given interval I, let $f_I := f\chi_{\Gamma_R(1,I)}$. Using the decomposition (16) as in the proof of Theorem 3, we obtain

$$|f^\vee(\xi)|^2 \leq \sum_n \sum_{|I|=|J|=2^{-n}, I\sim J} |f_I^\vee(\xi) f_J^\vee(\xi)| + \sum_{I\in I_E} |f_I^\vee|^2, \tag{29}$$

where, I_E is a set of finitely overlapping intervals of length $\approx R^{-1/2}$. Using (29), we have for any $q \geq 2$

$$\begin{aligned} \|f^\vee\|^2_{L^q(\mu)} &\leq \sum_{n=1}^{\log(R^{1/2})} \sum_{|I|=|J|=2^{-n}, I\sim J} \|f_I^\vee f_J^\vee\|_{L^{q/2}(\mu)} + \sum_{I\in I_E} \|f_I^\vee\|^2_{L^q(\mu)} \\ &=: S_1 + S_2. \end{aligned} \tag{30}$$

Note that for each $I \in I_E$, the support of f_I, $\Gamma_R(1,I)$, is contained in a rectangle D of dimensions $C \times CR^{1/2} \times CR$. Hence $f_I^\vee = f_I^\vee * \varphi_D^\vee$. Using this and Hölder's inequality, we have

$$|f_I^\vee| \leq (|f_I^\vee|^q * |\varphi_D^\vee|)^{1/q} \|\varphi_D^\vee\|_1^{1-1/q} \lesssim (|f_I^\vee|^q * |\varphi_D^\vee|)^{1/q}.$$

Using this, Fubini's theorem and Hausdorff-Young, we obtain

$$\begin{aligned} \|f_I^\vee\|^q_{L^q(\mu)} &\leq \int |f_I^\vee(x)|^q (\mu * |\varphi_D^\vee|)(x)\mathrm{d}x \lesssim \|f_I\|^q_{q'} \|\mu * |\varphi_D^\vee|\|_\infty \\ &\lesssim \|f_I\|_2^q R^{\frac{3}{2}(\frac{q}{2}-1)} R^{2-\alpha/2}. \end{aligned} \tag{31}$$

In the last inequality, we used Lemma 4.1(I) and Hölder's inequality. Since the intervals in I_E are finitely overlapping (19) implies that

$$S_2 \lesssim R^{\frac{3}{2}+\frac{1-\alpha}{q}}.$$

This bound takes care of S_2 for any $q \geq 2$. In what follows, we obtain bounds for S_1 for $q = 2$ and $q \geq 4$, the remaining case follows from interpolation.

Case 1). $q = 2$.

As in the proof of Theorem 3, it suffices to prove that for each n and for each pair $I \sim J$, $|I| = |J| = 2^{-n}$,

$$\|f_I^\vee f_J^\vee\|_{L^1(\mu)} \lesssim R^{2-\alpha/2} \|f_I\|_2 \|f_J\|_2, \tag{32}$$

where the implicit constant is independent of I, J and R. Note that the union of the supports of f_I and f_J are contained in a rectangle of dimensions $CR \times CR2^{-n} \times CR2^{-2n}$. Hence $f_I * f_J$ is supported in a rectangle D of dimensions $2CR \times 2CR2^{-n} \times 2CR2^{-2n}$, the longest axis being in the direction e and the second longest axis being in the direction of f, say. Note that e is a light direction and f is tangent to the light cone at e. Like above, we have

$$\|f_I^\vee f_J^\vee\|_{L^1(\mu)} \leq \int |f_I^\vee(x) f_J^\vee(x)| (\mu * |\varphi_D^\vee|)(x) dx. \tag{33}$$

Let T_e be the Lorentz transformation (see, e.g., [10], [11]) satisfying

$$T_e(e) = e, \quad T_e(f) = 2^n f, \quad T_e(e \times f) = 2^{2n} e \times f.$$

Let $F_I(\xi) = f_I(T_e^{-1}(\xi))$ and $F_J(\xi) = f_J(T_e^{-1}(\xi))$. Note that F_I is supported in $\Gamma_R(2^{2n}, I')$ and F_J is supported in $\Gamma_R(2^{2n}, J')$, where I' and J' are intervals of length ≈ 1 and the distance between them is ≈ 1. Also note that

$$\begin{aligned} f_I^\vee(x) &= F_I^\vee(T_e^{-1}(x)) 2^{-3n}, \\ \|f_I\|_2 &= \|F_I\|_2 2^{-3n/2}. \end{aligned} \tag{34}$$

Substituting (34) in (33) and then changing the variable, we get

$$\|f_I^\vee f_J^\vee\|_{L^1(\mu)} \lesssim 2^{-3n} \int |F_I^\vee(u) F_J^\vee(u)| (\mu * |\varphi_D^\vee|)(T_e(u)) \mathrm{d}u. \tag{35}$$

Consider a tiling of $\mathbf{R}^3$ with boxes P of side length $100\ 2^{-2n}$. Note that (22) is valid for ϕ_P. Let $F_{I,P} := \widehat{f_I^\vee \phi_P}$. Using (22) in (35) and then Hölder's inequality, we obtain

$$\begin{aligned} \|f_I^\vee f_J^\vee\|_{L^1(\mu)} &\lesssim 2^{-3n} \sum_P \int |F_{I,P}^\vee(u) F_{J,P}^\vee(u)| (\mu * |\varphi_D^\vee|)(T_e(u)) \phi_P(u) \mathrm{d}u \\ &\lesssim 2^{-3n} \sum_P \left[\int |F_{I,P}^\vee(u) F_{J,P}^\vee(u)|^2 \mathrm{d}u \right]^{1/2} . \\ &\left[\int \left[(\mu * |\varphi_D^\vee|)(T_e(u)) \phi_P(u) \right]^2 \mathrm{d}u \right]^{1/2} . \end{aligned} \tag{36}$$

We estimate the first integral in (36) as in the proof of Theorem 3. Let $A_{I,P}$ be the support of $F_{I,P}$. By Parseval, Cauchy-Schwarz and Young's inequality

$$\begin{aligned} \int |F_{I,P}^\vee(u) F_{J,P}^\vee(u)|^2 \mathrm{d}u &= \int |F_{I,P} * F_{J,P}(\xi)|^2 \mathrm{d}\xi \\ &\lesssim \| |(\xi + A_{I,P}) \cap A_{J,P}| \|_{L^\infty(\xi)} \int (|F_{I,P}|^2 * |F_{J,P}|^2)(\xi) \mathrm{d}\xi \\ &\lesssim \| |(\xi + A_{I,P}) \cap A_{J,P}| \|_{L^\infty(\xi)} \|F_{I,P}\|_2^2 \, \|F_{J,P}\|_2^2. \end{aligned} \tag{37}$$

Like before $A_{I,P}$ is contained in $\mathrm{supp}(F_I) + \mathrm{supp}(\phi_P) = \mathrm{supp}(F_I) + P_{dual}$, where P_{dual} is a cube of side length $2^{2n}/100$ centered at the origin. Thus $A(I,P)$ is contained in $\Gamma_R(C2^{2n}, \frac{11}{10}I')$. The transversality of the cone (see, e.g., [10], [11]) implies that $\| |(\xi + A_{I,P}) \cap A_{J,P}| \|_{L^\infty(\xi)} \lesssim R2^{4n}$. Using this in (37), we see that

$$\int |F_{I,P}^\vee(u) F_{J,P}^\vee(u)|^2 \mathrm{d}u \lesssim R2^{4n} \|F_{I,P}\|_2^2 \|F_{J,P}\|_2^2. \tag{38}$$

Now, we obtain a bound for the second integral in (36).

$$\begin{aligned}\int \left[(\mu * |\varphi_D^\vee|)(T_e(u))\phi_P(u)\right]^2 \mathrm{d}u &\lesssim 2^{-3n}\|\mu * |\varphi_D^\vee|\|_\infty \int (\mu * |\varphi_D^\vee|)(u)\phi_P(T_e^{-1}(u))\mathrm{d}u \\ &= 2^{-3n}\|\mu * |\varphi_D^\vee|\|_\infty \int (\mu * |\varphi_D^\vee|)(u)\phi_{T_e(P)}(u)\mathrm{d}u \end{aligned} \tag{39}$$

By Lemma 4.1(I), we have

$$\|\mu * |\varphi_D^\vee|\|_\infty \lesssim R^{3-\alpha} 2^{n\alpha - 2n}. \tag{40}$$

Note that $T_e(P)$ has dimensions $C2^{-2n} \times C2^{-n} \times C$ and it is a multiple of a dual of D. Thus we can apply Lemma 4.1(III) to obtain

$$\int (\mu * |\varphi_D^\vee|)(u)\phi_{T_e(P)}(u)\mathrm{d}u \lesssim 2^{n - n\alpha}. \tag{41}$$

Using (40) and (41) in (39), we have

$$\int \left[(\mu * |\varphi_D^\vee|)(T_e(u))\phi_P(u)\right]^2 \mathrm{d}u \lesssim R^{3-\alpha} 2^{-4n}. \tag{42}$$

Substituting (38) and (42) into (36) and then using (34), we obtain (32):

$$\begin{aligned}\|f_I^\vee f_J^\vee\|_{L^1(\mu)} &\lesssim R^{2-\alpha/2} 2^{-3n} \sum_P \|F_{I,P}\|_2 \|F_{J,P}\|_2 \\ &\lesssim R^{2-\alpha/2} 2^{-3n} \left[\sum_P \|F_{I,P}\|_2^2\right]^{1/2} \left[\sum_P \|F_{J,P}\|_2^2\right]^{1/2} \\ &\lesssim R^{2-\alpha/2} 2^{-3n} \|F_I\|_2 \|F_J\|_2 = R^{2-\alpha/2} \|f_I\|_2 \|f_J\|_2.\end{aligned}$$

Case 2). $q \geq 4$.

Like above, we have

$$\|f_I^\vee f_J^\vee\|_{L^{q/2}(\mu)}^{q/2} \leq \int |f_I^\vee(x) f_J^\vee(x)|^{q/2} (\mu * |\varphi_D^\vee|)(x) dx, \tag{43}$$

where D is a rectangle of dimensions $CR \times CR2^{-n} \times CR2^{-2n}$. Using (40), we have

$$\begin{aligned}\|f_I^\vee f_J^\vee\|_{L^{q/2}(\mu)}^{q/2} &\lesssim R^{3-\alpha} 2^{n\alpha - 2n} \int |f_I^\vee(x) f_J^\vee(x)|^{q/2} \mathrm{d}x \\ &\lesssim R^{3-\alpha} 2^{n\alpha - 2n} \|f_I^\vee f_J^\vee\|_\infty^{\frac{q}{2}-2} \int |f_I^\vee(x) f_J^\vee(x)|^2 \mathrm{d}x\end{aligned} \tag{44}$$

Using the Lorentz transformation T_e as in case 1, we have

$$\int |f_I^\vee(x) f_J^\vee(x)|^2 \mathrm{d}x \lesssim R2^n \|f_I\|_2^2 \|f_J\|_2^2. \tag{45}$$

We also have

$$\|f_I^\vee f_J^\vee\|_\infty \le \|f_I * f_J\|_1 \le \|f_I\|_1 \|f_J\|_1 \lesssim R^2 2^{-n} \|f_I\|_2 \|f_J\|_2. \tag{46}$$

Substituting (45) and (46) into (44), we have

$$\|f_I^\vee f_J^\vee\|_{L^{q/2}(\mu)} \lesssim R^{2-\frac{2\alpha}{q}} 2^{n\left(\frac{2}{q}(\alpha+1)-1\right)} \|f_I\|_2 \|f_J\|_2. \tag{47}$$

Note that if $\alpha \ge \frac{q}{2} - 1$, then $\frac{2}{q}(\alpha+1) - 1 \ge 0$ and hence $R^{2-\frac{2\alpha}{q}} 2^{n\left(\frac{2}{q}(\alpha+1)-1\right)} \lesssim R^{\frac{3}{2}-\frac{\alpha-1}{q}}$. Otherwise, $R^{2-\frac{2\alpha}{q}} 2^{n\left(\frac{2}{q}(\alpha+1)-1\right)} \lesssim R^{2-\frac{2\alpha}{q}}$ Combining these two cases, we have

$$\|f_I^\vee f_J^\vee\|_{L^{q/2}(\mu)} \lesssim R^{\max(2-\frac{2\alpha}{q}, \frac{3}{2}-\frac{\alpha-1}{q})} \|f_I\|_2 \|f_J\|_2. \tag{48}$$

Substituting (48) into (30) yields the required bound. □

5. Appendix

In this appendix, we prove (8) and Theorem 1.

Take a non-negative Schwartz function φ supported in $B(0,2)$ in $\mathbf{R}^3$ such that

$$\widehat{\varphi} \ge 0 \quad \text{and } \widehat{\varphi}(\xi) \in [1/2, 1], \text{ for } |\xi| < 1.$$

Let $S = \{x \in \mathbf{R}^3 : x_3 \in [-2R, 2R], |x_1 - x_3| < 1, |x_2| < R^{1/2}\}$.

Using appropriate dilations one obtains a function φ_R with L^1 norm 1 and supported in a rectangle of dimensions $C \times CR^{-1/2} \times CR^{-1}$, and $\widehat{\varphi_R} \in [1/2, 1]$ in S. Let $\mathrm{d}\mu(x) = \varphi_R(x)\mathrm{d}x$. Note that

$$\int_\Gamma |\widehat{\mu}(Ru)|^2 \mathrm{d}\nu_\Gamma(u) \gtrsim R^{-1/2}. \tag{49}$$

Also note that

$$I_\alpha(\mu) \approx \int \frac{\widehat{\varphi_R}(\xi)^2}{|\xi|^{3-\alpha}} \mathrm{d}\xi \lesssim \begin{cases} 1 & , \alpha \in (0,1] \\ R^{\frac{\alpha-1}{2}} & , \alpha \in [1,2] \\ R^{\alpha-3/2} & , \alpha \in [2,3]. \end{cases} \tag{50}$$

The last inequality is easy to prove if one replaces $\widehat{\varphi_R}$ with the characteristic function of S. The inequality follows from the Schwartz decay of $\widehat{\varphi_R}$ away from S.

Note that (49) and (50) imply (8) for $\alpha \ge 1/2$.

To see that $\beta(\alpha) \le \alpha$, one may use the functions $f_\eta(x) = |x|^{-\eta}$. Let $\mathrm{d}\mu(x) = f_{3-\alpha_1/2}(x)\varphi(x)\mathrm{d}x$. Then

$$\widehat{\mu}(\xi) = c f_{\alpha_1/2} * \widehat{\varphi}(\xi) \approx (1 + |\xi|)^{-\alpha_1/2}.$$

Thus $I_\alpha(\mu) \approx 1$ and $\int_\Gamma |\widehat{\mu}(Ru)|^2 \mathrm{d}\nu(u) \approx R^{-\alpha_1}$, which imply that $\beta(\alpha) \leq \alpha$.

Proof of Theorem 1. [6] The following calculation yields the first bound. It suffices to consider the case $\alpha \leq a$.

$$\int |\widehat{\mu}(Ru)|^2 \mathrm{d}\nu(u) = \int \widehat{\nu}(R(x-y)) \mathrm{d}\mu(x)\mathrm{d}\mu(y) \lesssim \int \frac{\mathrm{d}\mu(x)\mathrm{d}\mu(y)}{(R|x-y|)^a}$$
$$\lesssim \int \frac{\mathrm{d}\mu(x)\mathrm{d}\mu(y)}{(R|x-y|)^\alpha} = R^{-\alpha} I_\alpha(\mu).$$

Second bound follows from the uncertainty principle. Let ϕ be a non-negative Schwartz function supported in $B(0,2)$ and $\phi(x) = 1$ for $x \in B(0,1)$. Using $\mathrm{d}\mu(x) = \phi(x)\mathrm{d}\mu(x)$, we get

$$\widehat{\mu}(u) = \widehat{\mu} * \widehat{\phi}(u).$$

Using Cauchy-Schwarz, we obtain

$$\begin{aligned} \int |\widehat{\mu}(Ru)|^2 \mathrm{d}\nu(u) &= \int \left| \int \widehat{\mu}(\xi)\widehat{\phi}(Ru-\xi)\mathrm{d}\xi \right|^2 \mathrm{d}\nu(u) \\ &\lesssim \|\widehat{\phi}\|_{L^1} \int |\widehat{\mu}(\xi)|^2 |\widehat{\phi}(Ru-\xi)| \mathrm{d}\xi \mathrm{d}\nu(u) \\ &\lesssim I_\alpha(\mu) \sup_\xi \left(|\xi|^{n-\alpha} \int |\widehat{\phi}(Ru-\xi)| \mathrm{d}\nu(u) \right). \end{aligned} \tag{51}$$

Note that the Schwartz decay, $|\widehat{\phi}(x)| \leq C_M(1+|x|)^{-M}$, and the density assumption on ν imply that

$$\int |\widehat{\phi}(Ru-\xi)| \mathrm{d}\nu(u) \lesssim \left\{ \begin{array}{ll} R^{-b} & , |\xi| \lesssim R \\ |\xi|^{-M} & , |\xi| >> R \end{array} \right\} \lesssim R^{d-\alpha-b} |\xi|^{\alpha-d}, \tag{52}$$

if M has been chosen large enough. Substituting (52) in (51) yields the second bound. $\square$

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Department of Mathematics, University of California, Berkeley, CA 94720-3840