QUASI-LINEAR DYNAMICS IN NONLINEAR SCHRÖDINGER EQUATION WITH PERIODIC BOUNDARY CONDITIONS

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ABSTRACT. It is shown that a large subset of initial data with finite energy (L^2 norm) evolves nearly linearly in nonlinear Schrödinger equation with periodic boundary conditions. These new solutions are not perturbations of the known ones such as solitons, semiclassical or weakly linear solutions.

1. INTRODUCTION

The nonlinear Schrödinger (NLS) equation

$$(1) \qquad iq_t + \Delta q + |q|^2 q = 0,$$

where $q : \mathbb{R}_t \times M_x \to \mathbb{C}$, frequently appears as the leading approximation of the envelope dynamics of a quasi-monochromatic plane wave propagating in a weakly nonlinear dispersive medium. It arises in a number of physical models in the description of nonlinear waves such as the propagation of a laser beam in a medium whose index of reflection is sensitive to the wave amplitude.

NLS has been considered on various domains such as $M = \mathbb{R}^n, \mathbb{T}^n$, with periodic or Dirichlet boundary conditions. One dimensional cubic NLS is integrable [17] and the explicit (or approximately explicit) solutions can be obtained as solitons, cnoidal waves, and their perturbations. There have been also many interesting results on the long time asymptotics of solutions of integrable NLS in the limit of small dispersion, see *e.g.* the recent monograph [10], [6, 16, 3] and references therein.

Recent results in optical communication literature (see, *e.g.* [2, 7, 13], and the appendix) suggest that for some initial data (highly localized pulses) the evolution is nearly linear. Based on these studies, we introduce a large class of solutions, which we call quasi-linear, for one dimensional cubic NLS with periodic boundary conditions. These solutions can be characterized by the magnitude of Fourier coefficients of the initial data. We prove that these solutions evolve nearly linearly using a normal form reduction and estimates on Fourier sums. Although we do not explicitly use integrability, we do rely

on the integrability of the quartic normal form which is partially responsible for quasi-linear behavior. Therefore, similar results can be obtained for some nonlinear PDEs, such as $iq_t + q_{xxxx} + |q|^2 q = 0$, for which there are no integrability results. We do not study long time asymptotics but rather the finite time dynamics in the limit of spectral broadening of initial data. This broadening forces $\|q(x,0)\|_{H^s}$ to grow to infinity, making the analysis rather nontrivial even for the finite time interval. While, we consider the focusing case, our result holds for defocusing case as well. The reader will be able to see that our proof can be immediately adapted for the defocusing case, since nowhere our arguments rely on the nonlinearity sign.

In many engineering and physics applications, nonlinearity is unavoidable while modeling and optimizing a linear behavior is much easier than a nonlinear one. Therefore, it is an important question whether a nonlinear system can be made to behave linearly. In applied mathematics and physics literature, such a behavior has been observed in *e.g.* [1, 7, 8, 14, 15]. We believe that our result gives a systematic way to analyze this behavior in nonlinear systems when the energy is distributed over many Fourier harmonics.

2. Main Results

We consider the nonlinear Schrödinger equation with periodic boundary conditions,
$$iq_t + q_{xx} + 2|q|^2 q = 0,$$
with initial data in $q(0) \in L^2(-\pi, \pi)$. In [4], Bourgain proved the L^2 global well-posedness of this equation. The numerical simulations of quasi-linear regime for light wave communication systems suggest that the following statement should hold (see, *e.g.*, [7, 14])

Observation 1. *Assume that initial data is a localized Gaussian*
$$q(x,0) = \frac{1}{\sqrt{\varepsilon}}\, e^{-\frac{x^2}{\varepsilon^2}}\, h(x),$$
where $h(x)$ is a smooth cutoff near $x = \pm\pi/2$. Then the initial data evolves quasi-linearly,
$$\|q(x,t) - e^{it(\Delta + 4P)} q(x,0)\|_2 \to 0, \tag{2}$$
as $\varepsilon \to 0$ and for $t \le T$, where T is a fixed positive number, and $P = \|q(\cdot,0)\|_2^2/2\pi$.

We will prove (2) for a large class of initial data (including the ones above) characterized by the magnitude of Fourier coefficients. We will use Fourier transform in the form
$$q(x,t) = \sum_{n \in \mathbb{Z}} u(n,t) e^{inx}$$
$$u(m,t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} q(x,t) e^{-imx} dx,$$

so that the NLS equation takes the form

$$(3) \quad i\frac{du(m)}{dt} - m^2 u(m) + 2 \sum_{m_1+m_2-m_3=m} u(m_1)u(m_2)\bar{u}(m_3) = 0.$$

Our main result is the following theorem.

Theorem 2.1. *Let $P > 0$ and $C > 0$ be fixed. Assume that the Fourier sequence of the initial data $u(n,0) = \widehat{q(\cdot,0)}(n)$ satisfies*

$$\|u(\cdot,0)\|_{\ell^\infty} \leq C\varepsilon^{\frac{1}{2}}, \qquad \|u(\cdot,0)\|_{\ell^1} \leq C\varepsilon^{-\frac{1}{2}},$$

for sufficiently small $\varepsilon \in (0,1)$. Then, for each $t > 0$,

$$(4) \qquad \|q(\cdot,t) - e^{it(\Delta+4P)}q(\cdot,0)\|_{L^2} \lesssim \langle t\rangle\, \varepsilon^{1-},$$

where $P = \|q(\cdot,0)\|_2^2/2\pi$, $\langle t\rangle = \sqrt{1+t^2}$ and the implicit constant depends only on C.

Remark 2.1. *The initial data in the observation above satisfies the hypothesis of the theorem. In fact, if f is an H^s function for some $s > 1$ with compact support on $(-\pi,\pi)$, then*

$$f_\varepsilon(x) = \frac{1}{\sqrt{\varepsilon}}f(x/\varepsilon)$$

satisfies the hypothesis of the theorem.

By continuous dependence on initial data in L^2, it suffices to prove (4) for any $\delta > 0$ and for any initial data in the following subset of L^2:

$$B_{\varepsilon,C}^\delta = \{f \in L^2 : \|\hat{f}\|_{\ell^{p,\delta}} := \left[\sum_{n=-\infty}^\infty |\hat{f}(n)|^p e^{\delta|n|p}\right]^{1/p} \leq C\varepsilon^{\frac{1}{2}-\frac{1}{p}}, p \in [1,\infty]\}.$$

Since $B_{\varepsilon,C}^\delta \subset H^1$, we can introduce the Hamiltonian [11]

$$H(u) = i\sum_n n^2|u(n)|^2 - i\sum_{l(n)=0} u(n_1)u(n_2)\bar{u}(n_3)\bar{u}(n_4),$$

with conjugated variables $\{u(n), \bar{u}(n)\}_{n\in\mathbb{Z}}$, where $l(n) = n_1 + n_2 - n_3 - n_4$. The Hamiltonian flow is then given by

$$\dot{u}(n) = \frac{\partial H}{\partial \bar{u}(n)}.$$

Theorem 2.1 follows from the following by continuous dependence on initial data in L^2.

Theorem 2.2. *Let $P > 0$ and $C > 0$ be fixed. Assume that $\|q(0)\|_2^2 = 2\pi P$, and $q(\cdot,0) \in B_{\varepsilon,C}^\delta$ for some $\delta > 0$, and for sufficiently small $\varepsilon \in (0,1)$. Then, for each $t > 0$,*

$$(5) \qquad \|q(\cdot,t) - e^{it(\Delta+4P)}q(\cdot,0)\|_2 \lesssim \langle t\rangle\, \varepsilon^{1-},$$

where the implicit constant depends only on C.

The proof of Theorem 2.2 is based on the normal form transformations, see, e.g., [11], [12] and [5]. In Section 3, we introduce a canonical transformation $u = u(v)$ in the Fourier space which brings the equation into the form[1], see (15) and (16) below,

$$(6) \qquad\qquad \dot{v}(n) = i(n^2 + 4P)v(n) + E(v)(n).$$

We prove that the transformation $u = u(v)$ is near-identical in the following sense.

Proposition 2.1. *If $u \in B_{\varepsilon,C}^{\delta}$ or $v \in B_{\varepsilon,C}^{\delta}$, then*

$$\|u\|_{\ell^2} = \|v\|_{\ell^2}, \quad \text{and} \quad \|u - v\|_{\ell^{p,\delta}} \lesssim \varepsilon^{\frac{3}{2} - \frac{1}{p} -}$$

for $1 \le p \le \infty$, where the implicit constant depends on C and p.
In particular, if ε is sufficiently small, then $u \in B_{\varepsilon,C}^{\delta}$ implies $v \in B_{\varepsilon,2C}^{\delta}$ and vice versa.

Then, we estimate the error term $E(v)$ as follows

Proposition 2.2. *If $v \in B_{\varepsilon,C}^{\delta}$, then the error term $E(v)$ in the transformed equation (6) satisfies*

$$\|E(v)\|_{\ell^{p,\delta}} \lesssim \varepsilon^{\frac{3}{2} - \frac{1}{p} -},$$

for $1 \le p \le \infty$, where the implicit constant depends on C and p.

Propositions 2.1 and 2.2 imply Theorem 2.2. Indeed, assume that $q(\cdot, 0) \in B_{\varepsilon,C}^{\delta}$ for some $\delta > 0$, $C > 0$, and for sufficiently small $\varepsilon \in (0, 1)$. Multiplying (6) with $e^{-i(n^2 + 4P)t}$ and integrating over t, we obtain

$$v(n, t)e^{-i(n^2 + 4P)t} - v(n, 0) = \int_0^t e^{-i(n^2 + 4P)\tau} E(v) d\tau.$$

This and Propositions 2.1 and 2.2 imply, for each $p \in [1, \infty]$, that

$$(7) \qquad \|v(t) - e^{iLt}v(0)\|_{\ell^{p,\delta}} = \|v(t)e^{-iLt} - v(0)\|_{\ell^{p,\delta}} \lesssim t\,\varepsilon^{\frac{3}{2} - \frac{1}{p} -},$$

where $L(v)(n) = (n^2 + 4P)v(n)$. Finally, Proposition 2.1 and (7) imply, for $p \in [1, \infty]$, that

$$\|u(t) - e^{iLt}u(0)\|_{\ell^{p,\delta}} \le \|u(t) - v(t)\|_{\ell^{p,\delta}} + \|v(t) - e^{iLt}v(0)\|_{\ell^{p,\delta}}$$
$$+ \|e^{iLt}v(0) - e^{iLt}u(0)\|_{\ell^{p,\delta}}$$
$$\lesssim \langle t \rangle \, \varepsilon^{\frac{3}{2} - \frac{1}{p} -},$$

where the implicit constant depends on C. In particular, this yields the assertion of Theorem 2.2 as follows

$$\|q(t) - e^{it(\Delta + 2P)}q(0)\|_2 = \|u(t) - e^{iLt}u(0)\|_{\ell^2} \le \|u(t) - e^{iLt}u(0)\|_{\ell^{2,\delta}} \lesssim \langle t \rangle \, \varepsilon^{1-}.$$

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[1]Similar quasi-linear behavior can be obtained for the nonintegrable NLS $iq_t + q_{xxxx} + |q|^2 q = 0$ with the leading behavior given by $\dot{v}(n) = i(n^4 + 4P)v(n)$.

Notation.
We will frequently use convolution with $1/|n|$, which will be denoted by $\rho(n) = \frac{1}{|n|}\chi_{\mathbb{Z}\setminus\{0\}}(n)$. and we will also use the notation $\langle n \rangle = \sqrt{1+n^2}$.
We always assume by default that the summation index avoids the terms with vanishing denominators.
To avoid using unimportant constants, we will use $\lesssim$ sign:
$A \lesssim B$ means there is an absolute constant K such that $A \leq KB$. In some cases the constant will depend on parameters such as p.
$A \lesssim B(\eta-)$ means that for any $\gamma > 0$, $A \leq C_\gamma B(\eta - \gamma)$.
$A \lesssim B(\eta+)$ is defined similarly.

3. NORMAL FORM CALCULATIONS

Consider the change of variables $u_n \to v_n$, generated by the time 1 flow of a purely imaginary Hamiltonian F. Namely, solve

$$\frac{dw}{ds} = \frac{\partial F}{\partial \bar{w}}, \quad w|_{s=0} = v,$$

thus producing a symplectic transformation $u = u(v) := w|_{s=1}$. Let X_F^s be the time s map of the flow of F. Using Taylor expansion [11, 12], we have

$$(8) \quad H \circ X_F^1(v) = H(v) + \{H, F\}(v) + \ldots + \frac{1}{k!}\{\ldots\{\{H, \underbrace{F\}, F\}, \ldots, F}_{k}\}(v)$$

$$+ \int_0^1 \frac{(1-s)^k}{k!}\{\ldots\{\{H, \underbrace{F\}, F\}, \ldots, F}_{k+1}\} \circ X_F^s(v)\, ds,$$

where

$$(9) \qquad \{A, B\} = \sum_n \left(\frac{\partial A}{\partial u(n)} \frac{\partial B}{\partial \bar{u}(n)} - \frac{\partial A}{\partial \bar{u}(n)} \frac{\partial B}{\partial u(n)} \right)$$

is the Poisson bracket.

Recall that H has a quadratic and a quartic part

$$(10) \qquad\qquad\qquad H = \Lambda_2 + H_4,$$

where

$$(11) \qquad\qquad\qquad \Lambda_2 = i \sum m^2 |u(m)|^2.$$

We write $H_4 = H_4^{\mathrm{nr}} + H_4^{\mathrm{r}}$, where the superscripts "nr" and "r" denotes the non-resonant and resonant terms:

$$H_4^{\mathrm{nr}} = i \sum_{l(m)=0,\, q(m)\neq 0} v(m_1)v(m_2)\bar{v}(m_3)\bar{v}(m_4)$$

$$H_4^{\mathrm{r}} = i \sum_{l(m)=0,\, q(m)=0} v(m_1)v(m_2)\bar{v}(m_3)\bar{v}(m_4),$$

where $l(m) = m_1 + m_2 - m_3 - m_4$ and $q(m) = m_1^2 + m_2^2 - m_3^2 - m_4^2$. As usual H_4^{r} is the part of the Hamiltonian that commutes with Λ_2. Note that we can further decompose H_4^{r} as

$$H_4^{\mathrm{r}} = -i \sum_m |v(m)|^4 + 2i \sum_{m_1, m_2} |v(m_1)|^2 |v(m_2)|^2 := H_4^{\mathrm{r}1} + H_4^{\mathrm{r}2}.$$

We sequentially apply two normal form transformations generated by F_1 and F_2. We choose F_1 so that the following cancellation property holds

(12) $$\{\Lambda_2, F_1\} = -H_4^{\mathrm{nr}}.$$

We will prove that F_1 commutes with $H_4^{\mathrm{r}2}$. Using these cancellation properties in (8) with $k = 2$, we obtain

$$H \circ X_{F_1}^1 = \Lambda_2 + H_4^{\mathrm{r}1} + H_4^{\mathrm{r}2} + \{H_4^{\mathrm{r}1}, F_1\} + \frac{1}{2}\{H_4^{\mathrm{nr}}, F_1\} + \frac{1}{2}g_{F_1}^2(H_4)$$

$$+ \int_0^1 \frac{(1-s)^2}{2} g_{F_1}^3(H) \circ X_{F_1}^s \, ds,$$

where we used the notation

$$g_F^0(H) = H, \qquad g_F^{k+1}(H) = \{g_F^k, F\}, \quad k = 0, 1, 2, \ldots$$

Now, we apply the second transformation[2] generated by F_2 to eliminate the non-resonant terms in $\frac{1}{2}\{H_4^{\mathrm{nr}}, F_1\}$, i.e.,

(13) $$\{\Lambda_2, F_2\} = -\frac{1}{2}\{H_4^{\mathrm{nr}}, F_1\}^{\mathrm{nr}}.$$

We will also prove that F_2 commutes with $H_4^{\mathrm{r}2}$. Using these cancellation properties as above in (8) (with $k = 1$), we obtain

$$H \circ X_{F_1}^1 \circ X_{F_2}^1 = \Lambda_2 + H_4^{\mathrm{r}2} + R,$$

where

$$R = H_4^{\mathrm{r}1} + \{H_4^{\mathrm{r}1}, F_1\} + \frac{1}{2}\{H_4^{\mathrm{nr}}, F_1\}^{\mathrm{r}} + \{H_4^{\mathrm{r}1}, F_2\} + \{\{H_4^{\mathrm{r}1}, F_1\}, F_2\} + K$$

$$+ \frac{1}{2}\{\{H_4^{\mathrm{nr}}, F_1\}, F_2\} + \{K, F_2\} + \int_0^1 (1-s) g_{F_2}^2(H \circ X_{F_1}^1) \circ X_{F_2}^s \, ds,$$

where

$$K = \frac{1}{2}g_{F_1}^2(H_4) + \int_0^1 \frac{(1-s)^2}{2} g_{F_1}^3(H) \circ X_{F_1}^s \, ds.$$

The transformed evolution equation is given by

(14) $$\dot{v}(n) = \frac{\partial (H \circ X_{F_1}^1 \circ X_{F_2}^1)}{\partial \bar{v}}.$$

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[2]It turns out that the transform generated by F_1 is not enough since the term $\{H_4^{\mathrm{nr}}, F_1\}$ is present in the Hamiltonian. The direct estimate of this term produces finite order nonlinear effect (see Subsection 4.4).

Note that contribution of the "leading" terms, $\Lambda_2 + H_4^{r2}$, is given by

$$\frac{\partial}{\partial \bar{v}(n)}\left(i\sum m^2|v(m)|^2 + 2i\sum_{m_1,m_2}|v(m_1)|^2|v(m_2)|^2\right) = i(n^2 + 4P)v(n).$$

Therefore, we can rewrite (14) as

(15) $$\dot{v}(n) = i(n^2 + 4P)v(n) + E(v)(n),$$

where

(16) $$E(v)(n) = \frac{\partial R}{\partial \bar{v}(n)}.$$

3.1. **Calculation of F_1 and F_2.** To obtain (12), we take F_1 of the form

$$F_1 = \sum_{l(m)=0} f(m_1, m_2, m_3, m_4)v(m_1)v(m_2)\bar{v}(m_3)\bar{v}(m_4).$$

We have

$$\{\Lambda_2, F_1\} = i\sum_m m^2\left(\bar{v}(m)\frac{\partial F_1}{\partial \bar{v}(m)} - v(m)\frac{\partial F_1}{\partial v(m)}\right)$$

$$= i\sum_{l(m)=0}(m_1^2+m_2^2-m_3^2-m_4^2)f(m_1,m_2,m_3,m_4)v(m_1)v(m_2)\bar{v}(m_3)\bar{v}(m_4).$$

Therefore, we let

(17) $$F_1 = \sum_{l(m)=0}\frac{v(m_1)v(m_2)\bar{v}(m_3)\bar{v}(m_4)}{m_1^2 + m_2^2 - m_3^2 - m_4^2} = \sum_{l(m)=0}\frac{v(m_1)v(m_2)\bar{v}(m_3)\bar{v}(m_4)}{2(m_1 - m_3)(m_2 - m_3)}.$$

Now, we calculate F_2. Using the Hamiltonian structure[3]

$$\frac{\partial H}{\partial \bar{v}(n)} = -\overline{\frac{\partial H}{\partial v(n)}}, \qquad \frac{\partial F_2}{\partial \bar{v}(n)} = -\overline{\frac{\partial F_2}{\partial v(n)}}$$

we obtain

$$\{H_4^{\mathrm{nr}}, F_1\}^{\mathrm{nr}} = 2i\sum_{\substack{m_4,m_5\neq m_1,\ m_2,m_3\neq m_6 \\ l(m)=0,\ q(m)\neq 0}}\frac{v(m_1)v(m_2)v(m_3)\bar{v}(m_4)\bar{v}(m_5)\bar{v}(m_6)}{(m_2 - m_6)(m_3 - m_6)} - c.c.$$

Therefore, a calculation similar to the one for F_1 yields

(18) $$F_2 = \sum_{\substack{m_4,m_5\neq m_1,\ m_2,m_3\neq m_6 \\ l(m)=0,\ q(m)\neq 0}}\frac{v(m_1)v(m_2)v(m_3)\bar{v}(m_4)\bar{v}(m_5)\bar{v}(m_6)}{q(m)(m_2 - m_6)(m_6 - m_3)} - c.c.$$

Here, $l(m) = m_1 + m_2 + m_3 - m_4 - m_5 - m_6$, and $q(m) = m_1^2 + m_2^2 + m_3^2 - m_4^2 - m_5^2 - m_6^2$.

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[3]These identities follow from the following easily checked ones: $\Re(H) = 0$, $\partial_v H(v,\bar{v}) + \partial_v \bar{H}(v,\bar{v}) = 0$ and $\partial_v \bar{H}(v,\bar{v}) = \overline{\partial_{\bar{v}} H(v,\bar{v})}$.

3.2. **Proof of Proposition 2.1.** First we state a simple corollary of Young's inequality. Recall that $\rho(n) = 1/|n|$ for $n \neq 0$ and $\rho(0) = 0$.

Lemma 3.1. *For any $p > 1$, for any choices of $\pm$ signs*

$$\big\| \sum_j w(\pm n \pm j)\rho(\pm j) \big\|_{\ell^p_n} \lesssim \|w\|_{\ell^{p-}}.$$

With some abuse of notation, we denote each sum of the above form by $w * \rho$.

Proof. Recall that by Young's inequality, $\|w * \rho\|_{\ell^p} \lesssim \|w\|_{\ell^q}\|\rho\|_{\ell^r}$, where $1 + \frac{1}{p} = \frac{1}{q} + \frac{1}{r}$. The lemma follows since $\rho \in \ell^q$ for any $q > 1$. $\square$

Proof of Proposition 2.1. First note that the equality of the ℓ^2 norms follows from Hamiltonian formalism. Indeed, it is straightforward to verify that $\{F, Q\} = 0$ (where $Q(u) = \|u\|_2^2$), which implies ℓ^2 norm conservation. To prove the second statement, we should estimate the time 1 map of the flow of F_1 and of F_2. We start with F_1.

$$(19) \qquad \frac{dw(n)}{ds} = \frac{\partial F_1}{\partial \bar{w}(n)} = \sum_{m_1+m_2-m_3-n=0} \frac{w(m_1)w(m_2)\bar{w}(m_3)}{(m_1-n)(m_2-n)}.$$

Multiplying with $e^{\delta|n|}$, we estimate (assuming that $w \in B^\delta_{\varepsilon,C}$)

$$\big| e^{\delta|n|} \frac{dw(n)}{ds} \big| \leq$$

$$\leq \sum_{m_1+m_2-m_3-n=0} \frac{e^{-\delta(|m_1|+|m_2|+|m_3|-|n|)}}{|m_1-n||m_2-n|} |w(m_1)e^{\delta|m_1|}w(m_2)e^{\delta|m_2|}w(m_3)e^{\delta|m_3|}|$$

$$\leq \|w\|_{\ell^{\infty,\delta}} \sum_{m_1,m_2} \frac{|w(m_1)|e^{\delta|m_1|}|w(m_2)|e^{\delta|m_2|}}{|m_1-n||m_2-n|} \leq \|w\|_{\ell^{\infty,\delta}}[|w|e^{\delta|\cdot|} * \rho]^2(n).$$

In the second line, we used the fact that $|m_1| + |m_2| + |m_3| - |n| \geq 0$. Therefore, by Lemma 3.1, we obtain

$$\big\| \frac{dw}{ds} \big\|_{\ell^{\infty,\delta}} \leq \|w\|_{\ell^{\infty,\delta}} \||w| \, e^{\delta|\cdot|} * \rho\|_{\ell^\infty}^2 \leq \|w\|_{\ell^{\infty,\delta}} \|w\|_{\ell^{q,\delta}}^2$$

for any $1 \leq q < \infty$. Similarly, using Lemma 3.1, we obtain

$$\big\| \frac{dw}{ds} \big\|_{\ell^{1,\delta}} \leq \|w\|_{\ell^{\infty,\delta}} \||w| \, e^{\delta|\cdot|} * \rho\|_2^2 \lesssim \|w\|_{\ell^{\infty,\delta}} \|w\|_{\ell^{2-,\delta}}^2.$$

The last two inequalities imply that if $w(0) \in B^\delta_{\varepsilon,C}$ (or $w(1) \in B^\delta_{\varepsilon,C}$) then

$$\|w(s) - w(0)\|_{\ell^{\infty,\delta}} \lesssim \varepsilon^{\frac{3}{2}-}, \qquad \|w(s) - w(0)\|_{\ell^{1,\delta}} \lesssim \varepsilon^{\frac{1}{2}-}.$$

This completes the proof for F_1. In the proof for F_2, we omit some of the details, in particular the multiplication with $e^{\delta|n|}$ argument above, since it works exactly in the same way. To estimate the ℓ^p norm of the right hand side of

$$\frac{dw(n)}{ds} = \frac{\partial F_2}{\partial \bar{w}(n)},$$

we use duality:

$$\left\|\frac{\partial F_2}{\partial \bar{w}(n)}\right\|_{\ell^p} = \sup_{\|h\|_{\ell^{p'}}=1} \left|\sum h(n)\frac{\partial F_2}{\partial \bar{w}(n)}\right|. \tag{20}$$

Note that the right hand side of (20) can be estimated by the sum of six terms of the form

$$\tilde{F}_2(w_1,\ldots,w_6) := \sum_{\substack{m_4,m_5\neq m_1,\ m_2,m_3\neq m_6 \\ l(m)=0,\ q(m)\neq 0}} \frac{w_1(m_1)\cdots w_6(m_6)}{|q(m)||m_2-m_6||m_6-m_3|}, \tag{21}$$

where in the jth term $w_j = |h|$ and the others are $|v|$. The required estimates for these terms follow by applying Lemma 3.2 below with arbitrarily small η and with $i = j$ if $p' = 1$ and with $k = j$ if $p' = \infty$. $\hfill\square$

Lemma 3.2. *For any $\eta > 0$ and for any distinct $i, k \in \{1, 2, 3, 4, 5, 6\}$, there is a permutation (i_1, i_2, i_3, i_4) of the remaining indices such that*

$$\tilde{F}_2(w_1,\ldots,w_6) \lesssim \|w_i\|_{\ell^1}\|w_k\|_{\ell^\infty}\|w_{i_1}\|_{\ell^1}\prod_{l=2}^{4}\|w_{i_l}\|_{\ell^\infty}^{\frac{1}{1+\eta}}\|w_{i_l}\|_{\ell^1}^{\frac{\eta}{1+\eta}}.$$

Proof. Fix $\eta > 0$, i, and k. By Holder's inequality we have

$$\tilde{F}_2 \leq \Big[\sum_{\substack{m_4,m_5\neq m_1,\ m_2,m_3\neq m_6 \\ l(m)=0,\ q(m)\neq 0}} \frac{w_1(m_1)\cdots w_6(m_6)}{|q(m)|^{1+\eta}|m_2-m_6|^{1+\eta}|m_6-m_3|^{1+\eta}}\Big]^{\frac{1}{1+\eta}} \tag{22}$$

$$\times \Big[\sum_{l(m)=0} w_1(m_1)\cdots w_6(m_6)\Big]^{\frac{\eta}{1+\eta}}.$$

The second line is bounded by

$$\|w_k\|_{\ell^\infty}^{\frac{\eta}{1+\eta}} \prod_{l=1,l\neq k}^{6} \|w_l\|_{\ell^1}^{\frac{\eta}{1+\eta}}.$$

The required estimate for the sum in the first line follows from the following claim: For any permutation (j_1, j_2, j_3) of $\{1, 4, 5\}$, and for any permutation (n_1, n_2, n_3) of $\{2, 3, 6\}$, we have

$$\sum_{\substack{m_4,m_5\neq m_1,\ m_2,m_3\neq m_6 \\ l(m)=0,\ q(m)\neq 0}} \frac{w_1(m_1)\cdots w_6(m_6)}{|q(m)|^{1+\eta}|m_2-m_6|^{1+\eta}|m_6-m_3|^{1+\eta}} \lesssim \tag{23}$$

$$\lesssim \|w_{j_1}\|_{\ell^\infty}\|w_{j_2}\|_{\ell^\infty}\|w_{j_3}\|_{\ell^1}\|w_{n_1}\|_{\ell^\infty}\|w_{n_2}\|_{\ell^\infty}\|w_{n_3}\|_{\ell^1}.$$

To prove this inequality, replace m_{j_1} in the sum with a linear combination of other indices using the identity $l(m) = 0$. We claim that

$$\||q(m)|^{-1-\eta}\|_{\ell^1_{m_{j_2}}} \lesssim 1,$$

where the implicit constant is independent of the remaining indices. Indeed, it suffices to consider the cases $j_1 = 1$, $j_2 = 4$ and $j_1 = 4$, $j_2 = 5$ since m_4 and m_5 enter symmetrically. In the former case

$$q(m) = (m_4 + m_5 + m_6 - m_2 - m_3)^2 + m_2^2 + m_3^2 - m_4^2 - m_5^2 - m_6^2$$
$$= C_1 m_4 + C_2,$$

where the integers C_1, C_2 depend on m_2, m_3, m_5, m_6. Moreover, $C_1 \neq 0$ since $m_1 \neq m_4$. Therefore,

$$\sup_{m_2, m_3, m_5, m_6} \||q(m)|^{-1-\eta}\|_{\ell^1_{m_4}} \lesssim 1.$$

In the latter case

$$q(m) = C_1 + C_2 m_5 - 2m_5^2,$$

where the integers C_1, C_2 depend on m_1, m_2, m_3, m_6. Since for any integers n, C_1, C_2, the equation $n = C_1 + C_2 m_5 - 2m_5^2$ has at most two solutions, we have

$$\sup_{m_1, m_2, m_3, m_6} \||q(m)|^{-1-\eta}\|_{\ell^1_{m_5}} \lesssim 1.$$

Using this claim, we obtain

$$(23) \lesssim \|w_{j_1}\|_{\ell^\infty} \|w_{j_2}\|_{\ell^\infty} \|w_{j_3}\|_{\ell^1} \sum \frac{w_2(m_2) w_3(m_3) w_6(m_6)}{|m_2 - m_6|^{1+\eta} |m_6 - m_3|^{1+\eta}}$$

$$\leq \|w_{j_1}\|_{\ell^\infty} \|w_{j_2}\|_{\ell^\infty} \|w_{j_3}\|_{\ell^1} \|w_{n_1}\|_{\ell^\infty} \|w_{n_2}\|_{\ell^\infty} \sum \frac{w_{n_3}(m_{n_3})}{|m_2 - m_6|^{1+\eta} |m_6 - m_3|^{1+\eta}}$$

$$\lesssim \|w_{j_1}\|_{\ell^\infty} \|w_{j_2}\|_{\ell^\infty} \|w_{j_3}\|_{\ell^1} \|w_{n_1}\|_{\ell^\infty} \|w_{n_2}\|_{\ell^\infty} \|w_{n_3}\|_{\ell^1}.$$

$$\square$$

3.3. Cancellation property of $H_4^{\mathrm{r}2}$.
We claim that $\{H_4^{\mathrm{r}2}, F_j\} = 0, j = 1, 2$. Indeed, by (17) and (18), both F_1 and F_2 have the phase invariant property

$$F_j(v) = F_j(ve^{i\phi}),$$

but the evolution induced by $H_4^{\mathrm{r}2}$ is just uniform phase rotation,

$$v(n,t) = e^{i2Pt} v(n,0).$$

Thus,

$$\{H_4^{\mathrm{r}2}, F_j\} := \frac{d}{dt} F_j(X_{H_4^{\mathrm{r}2}}^{t=0}) = 0, \quad j = 1, 2.$$

4. Proof of Proposition 2.2

Assuming that $v \in B_{\varepsilon, C}^\delta$, we should prove that the $\ell^{p, \delta}$ norm of each of the summands in (16) is $\lesssim \varepsilon^{3/2 - 1/p -}$ for $p = 1$ and $p = \infty$. To simplify the exposition, we will do this only in the case $\delta = 0$. The proof for the case $\delta > 0$ is similar by using the simple multiplication by $e^{\delta|\cdot|}$ argument we used in the proof of Proposition 2.1.

Note that it suffices to consider the $\partial_{\bar{v}(k)}$ derivatives of the following terms

$$H_4^{\mathrm{r}1}, \quad \{H_4^{\mathrm{r}1}, F_1\}, \quad \{H_4^{\mathrm{r}1}, F_2\}, \quad \{H_4^{\mathrm{nr}}, F_1\}^{\mathrm{r}}, \quad g_{F_2}^b g_{F_1}^a(H_4), \quad a+b \geq 2,$$

and the terms involving integrals.

We define

$$f_1(v_1, v_2, v_3)(k) := \sum_{m_1, m_2 \neq k} \frac{v_1(m_1)v_2(m_2)v_3(m_1+m_2-k)}{(m_1-k)(m_2-k)}$$

so that $f_1(v, v, \bar{v})(k) = \partial_{\bar{v}(k)}F_1$. Similarly we define $f_2(v_1, v_2, v_3, v_4, v_5)(k)$ so that $f_2(v, v, v, \bar{v}, \bar{v})(k) = \partial_{\bar{v}(k)}F_2$. The following Lemma will be used repeatedly:

Lemma 4.1. *I) For any $q \in [1, \infty]$ and any permutation (i_1, i_2, i_3) of $(1, 2, 3)$, we have*

$$\|f_1(v_1, v_2, v_3)\|_{\ell^q} \lesssim \|v_{i_1}\|_{\ell^q} \|v_{i_2}\|_{\ell^{\infty-}} \|v_{i_3}\|_{\ell^{\infty-}}.$$

II) For any $q \in [1, \infty]$, for any $\eta > 0$, and for any $i \in \{1, 2, 3, 4, 5\}$ there is a permutation (i_1, i_2, i_3, i_4) of the set $\{1, 2, 3, 4, 5\} \setminus \{i\}$ such that

$$\|f_2(v_1, v_2, v_3, v_4, v_5)\|_{\ell^q} \lesssim \|v_i\|_{\ell^q} \|v_{i_1}\|_{\ell^1} \prod_{l=2}^4 \|v_{i_l}\|_{\ell^\infty}^{\frac{1}{1+\eta}} \|v_{i_l}\|_{\ell^1}^{\frac{\eta}{1+\eta}}.$$

Proof. Part I can easily be verified following the proof of Proposition 2.1 with $\delta = 0$. Part II follows from Lemma 3.2 and interpolation. $\qquad\square$

4.1. **Estimate of $\partial_{\bar{v}(k)}H_4^{\mathrm{r}1}$.** Recall that

$$H_4^{\mathrm{r}1} = i \sum_m |v(m)|^4,$$

and hence

$$\frac{\partial H_4^{\mathrm{r}1}}{\partial \bar{v}(k)} = 2i|v(k)|^2 v(k).$$

We estimate the contribution of this term as

$$\left\| \frac{\partial H_4^{\mathrm{r}1}}{\partial \bar{v}(\cdot)} \right\|_{\ell^\infty} \lesssim \|v^3\|_{\ell^\infty} \lesssim \varepsilon^{3/2},$$

and

$$\left\| \frac{\partial H_4^{\mathrm{r}1}}{\partial \bar{v}(\cdot)} \right\|_{\ell^1} \lesssim \|v^3\|_{\ell^1} = \|v\|_{\ell^3}^3 \lesssim \varepsilon^{\frac{3}{2}-1}.$$

4.2. **Estimates for $\partial_{\bar{v}(k)}\{H_4^{\mathrm{r}1}, F_1\}$ and $\partial_{\bar{v}(k)}\{H_4^{\mathrm{r}1}, F_2\}$.** Let

$$\tilde{H}_4^{\mathrm{r}1}(v_1, v_2, v_3, v_4) := \sum_n v_1(n)v_2(n)v_3(n)v_4(n).$$

We use duality as in (20). Note that $\sum_k |\partial_{\bar{v}(k)}\{H_4^{\mathrm{r}1}, F_1\}||h(k)|$ is bounded by the sum of the following two terms

$$\tilde{H}_4^{\mathrm{r}1}(|f_1(|v|, |v|, |v|)|, |h|, |v|, |v|), \quad \tilde{H}_4^{\mathrm{r}1}(|f_1(|h|, |v|, |v|)|, |v|, |v|, |v|),$$

and similar terms obtained by permuting the arguments. The following estimates (with $p = 1$ and $p = \infty$), which follow from the definition of $\tilde{H}_4^{\mathrm{r}1}$ and Lemma 4.1, completes the analysis of $\partial_{\bar{v}(k)}\{H_4^{\mathrm{r}1}, F_1\}$:

$$\tilde{H}_4^{\mathrm{r}1}(|f_1(|v|, |v|, |v|)|, |h|, |v|, |v|) \lesssim \|h\|_{\ell^{p'}} \|v\|_{\ell^p} \|v\|_{\ell^\infty} \|f_1(|v|, |v|, |v|)\|_{\ell^\infty}$$
$$\lesssim \|h\|_{\ell^{p'}} \|v\|_{\ell^p} \|v\|_{\ell^\infty}^2 \|v\|_{\ell^{\infty-}}^2 \lesssim \varepsilon^{\frac{5}{2} - \frac{1}{p}-}.$$

$$\tilde{H}_4^{\mathrm{r}1}(|f_1(|h|, |v|, |v|)|, |v|, |v|, |v|) \lesssim \|f_1(|h|, |v|, |v|)\|_{\ell^{p'}} \|v\|_{\ell^p} \|v\|_{\ell^\infty}^2$$
$$\lesssim \|h\|_{\ell^{p'}} \|v\|_{\ell^p} \|v\|_{\ell^\infty}^2 \|v\|_{\ell^{\infty-}}^2 \lesssim \varepsilon^{\frac{5}{2} - \frac{1}{p}-}.$$

We estimate $\partial_{\bar{v}(k)}\{H_4^{\mathrm{r}1}, F_2\}$ similarly. The estimates below imply the required bound

$$\tilde{H}_4^{\mathrm{r}1}(|f_2(|v|, \ldots, |v|)|, |h|, |v|, |v|) \lesssim \|h\|_{\ell^{p'}} \|v\|_{\ell^p} \|v\|_{\ell^\infty} \|f_2(|v|, \ldots, |v|)\|_{\ell^\infty}$$
$$\lesssim \|h\|_{\ell^{p'}} \|v\|_{\ell^p} \|v\|_{\ell^\infty}^2 \|v\|_{\ell^1} \|v\|_{\ell^\infty}^{\frac{3}{1+\eta}} \|v\|_{\ell^1}^{\frac{3\eta}{1+\eta}}$$
$$\lesssim \varepsilon^{\frac{5}{2} - \frac{1}{p}-}.$$

$$\tilde{H}_4^{\mathrm{r}1}(|f_2(|h|, |v|, |v|, |v|, |v|)|, |v|, |v|, |v|) \lesssim \|f_2(|h|, \ldots, |v|)\|_{\ell^{p'}} \|v\|_{\ell^p} \|v\|_{\ell^\infty}^2$$
$$\lesssim \|h\|_{\ell^{p'}} \|v\|_{\ell^1} \|v\|_{\ell^p} \|v\|_{\ell^\infty}^2 \|v\|_{\ell^\infty}^{\frac{3}{1+\eta}} \|v\|_{\ell^1}^{\frac{3\eta}{1+\eta}}$$
$$\lesssim \varepsilon^{\frac{5}{2} - \frac{1}{p}-}.$$

In both estimates, the last inequality is obtained by taking η sufficiently small.

4.3. Estimate of $\partial_{\bar{v}(k)}\{H_4^{\mathrm{nr}}, F_1\}^{\mathrm{r}}$.

Based on the calculations in Section 3.1, we have

$$\{H_4^{\mathrm{nr}}, F_1\}^{\mathrm{r}} = 2i \sum_{\substack{m_4, m_5 \neq m_1, \, m_2, m_3 \neq m_6 \\ l(m)=0, \, q(m)=0}} \frac{v(m_1)v(m_2)v(m_3)\bar{v}(m_4)\bar{v}(m_5)\bar{v}(m_6)}{(m_2 - m_6)(m_3 - m_6)} - c.c.$$

Using duality as above we need to estimate 6 terms of the form

$$(24) \qquad \sum_{\substack{m_4, m_5 \neq m_1, \, m_2, m_3 \neq m_6 \\ l(m)=0, \, q(m)=0}} \frac{v_1(m_1)v_2(m_2)v_3(m_3)v_4(m_4)v_5(m_5)v_6(m_6)}{|m_2 - m_6||m_3 - m_6|},$$

where $v_j = |h|$ and others are $|v|$. The required estimates follow from the following claim: For any $\eta > 0$, for any permutation (j_1, j_2, j_3) of $\{1, 4, 5\}$, and for any permutation (n_1, n_2, n_3) of $\{2, 3, 6\}$, we have

$$(24) \lesssim \|v_{j_1}\|_{\ell^\infty} \|v_{j_3}\|_{\ell^1} \|v_{n_1}\|_{\ell^\infty} \|v_{n_3}\|_{\ell^1} \left(\|v_{j_2}\|_{\ell^\infty} \|v_{n_2}\|_{\ell^\infty}\right)^{\frac{1}{1+\eta}} \left(\|v_{j_2}\|_{\ell^1} \|v_{n_2}\|_{\ell^1}\right)^{\frac{\eta}{1+\eta}}.$$

As in the proof of Lemma 3.2, see (22), the claim follows from an estimate for the following sum

$$(25) \qquad \sum_{\substack{m_4, m_5 \neq m_1, \ m_2, m_3 \neq m_6 \\ l(m)=0, \ q(m)=0}} \frac{v_1(m_1)_2 v(m_2) v_3(m_3) v_4(m_4) v_5(m_5) v_6(m_6)}{|m_2 - m_6|^{1+\eta} |m_3 - m_6|^{1+\eta}}.$$

First we replace j_1 in the equation $q(m) = 0$ using $l(m) = 0$. By symmetry it suffices to consider two cases $j_1 = 1$, $j_1 = 4$. In the former case we have

$$0 = (m_2 + m_3 - j_2 - j_3 - m_6)^2 + m_2^2 + m_3^2 - j_2^2 - j_3^2 - m_6^2$$
$$= -2j_2(m_2 + m_3 - j_3 - m_6) + (m_2 + m_3 - j_3 - m_6)^2 + m_2^2 + m_3^2 - j_3^2 - m_6^2.$$

Moreover, $m_2 + m_3 - j_3 - m_6 \neq 0$ since $m_1 \neq m_4, m_5$. Therefore, both j_1 and j_2 are determined by the remaining indices. This implies that

$$(25) \lesssim \|v_{j_1}\|_{\ell^\infty} \|v_{j_2}\|_{\ell^\infty} \|v_{j_3}\|_{\ell^1} \sum_{m_2, m_3 \neq m_6} \frac{v(m_2) v_3(m_3) v_6(m_6)}{|m_2 - m_6|^{1+\eta} |m_3 - m_6|^{1+\eta}}$$

$$\lesssim \|v_{j_1}\|_{\ell^\infty} \|v_{j_2}\|_{\ell^\infty} \|v_{j_3}\|_{\ell^1} \|v_{n_1}\|_{\ell^\infty} \|v_{n_2}\|_{\ell^\infty} \|v_{n_3}\|_{\ell^1},$$

which leads to the desired estimate as in the previous sections. The case $j_1 = 4$ is similar, the only difference is that j_2 is determined as roots of a quadratic polynomial instead of a linear one.

4.4. **Estimate of $\partial_{\bar{v}(k)} g_{F_2}^b g_{F_1}^a (H_4)$.** The bounds for $\partial_{\bar{v}(k)} g_{F_2}^b g_{F_1}^a (H_4)$ will be obtained inductively. Although we only need to consider the cases when $a + b \geq 2$, we start with the case $a = 1, b = 0$ for clarity. Note that $g_{F_1}^1 (H_4)$ is a sum of terms of the form

$$H_4(v_1, v_2, v_3, v_4) = \sum_{n_1 - n_2 + n_3 - n_4 = 0} v_1(n_1) v_2(n_2) v_3(n_3) v_4(n_4)$$

where one of v_i's is f_1 or $\bar{f}_1$ and the others are v or $\bar{v}$. To estimate $\left\| \partial_{\bar{v}(k)} g_{F_1}^1 (H_4) \right\|_{\ell^p}$, we use duality as before:

$$(26) \qquad \left\| \frac{\partial}{\partial \bar{v}(k)} g_{F_1}^1 (H_4) \right\|_{\ell^p} \leq \sup_{\|h\|_{p'}=1} \sum_k \left| \frac{\partial}{\partial \bar{v}(k)} g_{F_1}^1 (H_4) \right| |h(k)|.$$

Note that the sum in the right hand side of (26) is bounded by the sum of the following two terms

$$H_4(|f_1(|v|, |v|, |v|)|, |h|, |v|, |v|), \qquad H_4(|f_1(|h|, |v|, |v|)|, |v|, |v|, |v|)$$

and similar terms obtained by permuting the arguments. The following lemma will be used to estimate these terms and the ones appearing in the higher order commutators.

Lemma 4.2. *For any $q \in [1, \infty]$ and any permutation (i_1, i_2, i_3, i_4) of $(1, 2, 3, 4)$, we have*

$$|H_4(v_1, v_2, v_3, v_4)| \leq \|v_{i_1}\|_{\ell^q} \|v_{i_2}\|_{\ell^{q'}} \|v_{i_3}\|_{\ell^1} \|v_{i_4}\|_{\ell^1}.$$

Proof. Note that for any permutation we can write

$$H_4(v_1, v_2, v_3, v_4) = \sum_j v_{i_1}(j)\, v_{i_2} * v_{i_3} * v_{i_4}(j).$$

The statement follows from Hölder's and Young's inequalities. □

Using Lemma 4.2 and Lemma 4.1, we obtain

$$
\begin{aligned}
H_4(|f_1(|v|,|v|,|v|)|,|h|,|v|,|v|) &\lesssim \|h\|_{\ell^{p'}} \|v\|_{\ell^p} \|v\|_{\ell^1} \|f_1(|v|,|v|,|v|)\|_{\ell^1} \\
&\lesssim \|h\|_{\ell^{p'}} \|v\|_{\ell^p} \|v\|_{\ell^1} \|v\|_{\ell^1} \|v\|_{\ell^{\infty-}}^2 \\
&\lesssim \|h\|_{\ell^{p'}} \varepsilon^{\frac{1}{2}-\frac{1}{p}-}.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
H_4(|f_1(|h|,|v|,|v|)|,|v|,|v|,|v|) &\lesssim \|f_1(|h|,|v|,|v|)\|_{\ell^{p'}} \|v\|_{\ell^p} \|v\|_{\ell^1}^2 \\
&\lesssim \|h\|_{\ell^{p'}} \|v\|_{\ell^{\infty-}}^2 \|v\|_{\ell^p} \|v\|_{\ell^1}^2 \\
&\lesssim \|h\|_{\ell^{p'}} \varepsilon^{\frac{1}{2}-\frac{1}{p}-}.
\end{aligned}
$$

Similar bounds follow for the terms obtained by permuting the arguments. Therefore we have

$$\left\| \frac{\partial}{\partial \bar{v}(k)} g_{F_1}^1(H_4) \right\|_{\ell^p} \lesssim \varepsilon^{\frac{1}{2}-\frac{1}{p}-}.$$

Note that this gives an error of order 1 when $p = 2$. This explains why we consider higher order commutators and a second normal form transform (see footnote 2). This proof motivates the following generalization:

Lemma 4.3. *Consider $H_4(|v|,|v|,|v|,|v|)$. Repeatedly (a times) replace one of the v's with $f_1(|v|,|v|,|v|)$. Then repeatedly (b times) replace one of the v's with $f_2(|v|,|v|,|v|,|v|,|v|)$. Finally, replace one of the v's with h. We denote any such function by $H_{4,a,b}(f_1, f_2, h, v)$. Then, for $p = 1$ and $p = \infty$, we have*

$$|H_{4,a,b}(f_1, f_2, h, v)| \lesssim \|h\|_{\ell^{p'}} \varepsilon^{a+b-\frac{1}{2}-\frac{1}{p}-}.$$

Proof. First by using Lemma 4.1 repeatedly (with sufficiently small η, we see that any composition of f_1's and f_2's satisfy

$$(27) \qquad \| \cdot \|_{\ell^q} \lesssim \|v\|_{\ell^q} \|v\|_{\ell^{\infty-}}^{2a} \left[\|v\|_{\ell^1}^{1+} \|v\|_{\ell^\infty}^{3-} \right]^b,$$

where a is the number of f_1's and b is the number of f_2's appearing in the composition.

Now, note that H_4 has four arguments. Let a_j (resp. b_j) be the number of f_1's (resp. f_2's) appearing in the jth argument. Only one of the arguments contains h, say the first one. Using Lemma 4.2, we have

$$|H_4(v_1, v_2, v_3, v_4)| \lesssim \|v_1\|_{\ell^{p'}} \|v_2\|_{\ell^p} \|v_3\|_{\ell^1} \|v_4\|_{\ell^1}.$$

Using (27), we have

$$\|v_2\|_{\ell^p} \|v_3\|_{\ell^1} \|v_4\|_{\ell^1} \lesssim \|v\|_{\ell^p} \|v\|_{\ell^1} \|v\|_{\ell^1} \|v\|_{\ell^{\infty-}}^{2(a_2+a_3+a_4)} \left[\|v\|_{\ell^1}^{1+} \|v\|_{\ell^\infty}^{3-} \right]^{b_2+b_3+b_4}$$

Next, note that v_1 is either $|h|$ (in which case we stop) or $f_1(v_{1,1}, v_{1,2}, v_{1,3})$ or $f_2(v_{1,1}, \ldots, v_{1,5})$. In the latter cases, without loss of generality, $v_{1,1}$ contains $|h|$. We estimate, using (27) and a simple induction,

$$\|v_1\|_{\ell^{p'}} \lesssim \|h\|_{\ell^{p'}} \|v\|_{\ell^{\infty-}}^{2a_1} \left[\|v\|_{\ell^1}^{1+} \|v\|_{\ell^{\infty}}^{3-} \right]^{b_1}.$$

Combining these estimates we obtain

$$|H_{4,a,b}(f_1, f_2, h, v)| \lesssim \|h\|_{\ell^{p'}} \|v\|_{\ell^p} \|v\|_{\ell^1} \|v\|_{\ell^1} \|v\|_{\ell^{\infty-}}^{2a} \left[\|v\|_{\ell^1}^{1+} \|v\|_{\ell^{\infty}}^{3-} \right]^{b}$$

$$\lesssim \|h\|_{\ell^{p'}} \varepsilon^{a+b-\frac{1}{2}-\frac{1}{p}-}.$$

$\square$

Using duality as above we see that the right hand side of (26) for $\partial_{\bar{v}(k)} g_{F_2}^b g_{F_1}^a (H_4)$ can be bounded by a finite sum of functions $H_{4,a,b}(f_1, f_2, h, v)$. Therefore, Lemma 4.3 implies that

$$\left\| \frac{\partial g_{F_2}^b g_{F_1}^a (H_4)}{\partial \bar{v}(k)} \right\|_{\ell^p} \lesssim \varepsilon^{a+b-\frac{1}{2}-\frac{1}{p}-} \lesssim \varepsilon^{\frac{3}{2}-\frac{1}{p}-}, \quad \text{if } a+b \geq 2.$$

4.5. **Remainder Estimates.** It remains to estimate the terms involving integrals. Note that it suffices to prove the inequalities

$$\sup_{s \in [0,1]} \left\| \frac{\partial}{\partial \bar{v}(k)} g_{F_1}^3 (H) \circ X_{F_1}^s \right\|_{\ell^p} \lesssim \varepsilon^{\frac{3}{2}-\frac{1}{p}-},$$

$$\sup_{s \in [0,1]} \left\| \frac{\partial}{\partial \bar{v}(k)} \{ g_{F_1}^3 (H) \circ X_{F_1}^s, F_2 \} \right\|_{\ell^p} \lesssim \varepsilon^{\frac{3}{2}-\frac{1}{p}-},$$

$$\sup_{s \in [0,1]} \left\| \frac{\partial}{\partial \bar{v}(k)} g_{F_2}^2 (H \circ X_{F_1}^1) \circ X_{F_2}^s \right\|_{\ell^p} \lesssim \varepsilon^{\frac{3}{2}-\frac{1}{p}-}$$

for $p = 1, \infty$ assuming that $\|v\|_{\ell^p} \lesssim \varepsilon^{\frac{1}{2}-\frac{1}{p}}, p \in [1,\infty]$. Since we have to estimate the composite function derivative, we first study the bounds on the derivatives of $X_{F_j}^s(v)$, $j = 1, 2$, $s \in [0,1]$, more precisely, let $w(m) = [X_{F_j}^s(v)](m)$, which is the solution at $t = s$ of the system

$$\frac{dw(m)}{dt} = \frac{\partial F_j}{\partial \bar{w}(m)}, \quad w|_{t=0} = v.$$

Differentiating this equation with respect to initial condition $w(n)|_{t=0} = v(n)$ and using the notation D_n, we see that $\left| \frac{d}{dt} D_n w(m) \right|$ is bounded by a sum of terms of the form

$$f_1(v_1, v_2, v_3)(m), \quad \text{for } j = 1,$$

$$f_2(v_1, v_2, v_3, v_4, v_5)(m), \quad \text{for } j = 2,$$

where one of the v_k's is $|D_n w|$ and the others are $|w|$. Without loss of generality we can assume that $v_1 = |D_n w|$. We have a similar formula for

$\frac{d}{ds}D_n\bar{w}$. Note that at $s = 0$, we have

$$\left\|D_n w(m)\right\|_{\ell_m^\infty \ell_n^1} = \left\|D_n w(m)\right\|_{\ell_n^\infty \ell_m^1} = 1.$$

We will prove that both of these norms remain bounded for $s \in [0,1]$. Taking the $\ell_m^\infty \ell_n^1$ norm of f_j we obtain

$$\left\|\frac{d}{dt}D_n w(m)\right\|_{\ell_m^\infty \ell_n^1} \lesssim \left\|f_j(|D_n w|, \dots, |w|)(m)\right\|_{\ell_m^\infty \ell_n^1}$$
$$\leq \left\|f_j(\|D_n w\|_{\ell_n^1}, \dots, |w|)(m)\right\|_{\ell_m^\infty}$$
$$\lesssim \|D_n w(m)\|_{\ell_m^\infty \ell_n^1} \varepsilon^{1-}.$$

In the last line, we used Lemma 4.1 (for sufficiently small η). This implies that (with $w(m) = [X_{F_j}^s(v)](m)$, $j = 1,2$)

$$(28) \qquad \sup_{0 \leq s \leq 1} \left\|D_n w(m)\right\|_{\ell_m^\infty \ell_n^1} \lesssim 1.$$

Similarly, we obtain

$$(29) \qquad \sup_{0 \leq s \leq 1} \left\|D_n w(m)\right\|_{\ell_n^\infty \ell_m^1} \lesssim 1.$$

We also need the following estimates for the higher order derivatives of $w = X_{F_1}^s(v)$ with respect to the initial conditions:

$$\|D_j D_n w(k)\|_{\ell_{j,n}^\infty \ell_k^1} \lesssim \varepsilon^{\frac{1}{2}-}, \qquad \|D_j D_n w(k)\|_{\ell_{k,n}^\infty \ell_j^1} \lesssim \varepsilon^{\frac{1}{2}-},$$
$$\|D_j D_m D_n w(k)\|_{\ell_{j,m,n}^\infty \ell_k^1} \lesssim 1, \qquad \|D_j D_m D_n w(k)\|_{\ell_{k,m,n}^\infty \ell_j^1} \lesssim 1,$$

which can be obtained using Lemma 4.1 as in the proof of (28), (29).

Remark 4.1. *For $\delta > 0$, a similar argument implies*

$$\left\|e^{\delta|n-m|}D_n w(m)\right\|_{\ell_m^\infty \ell_n^1} \lesssim 1, \qquad \left\|e^{\delta|n-m|}D_n w(m)\right\|_{\ell_n^\infty \ell_m^1} \lesssim 1,$$

and for higher order derivatives of $w = X_{F_1}^s(v)$ we have

$$\left\|e^{\delta|j_1+\cdots+j_k-m|}D_{j_1}\dots D_{j_k}w(m)\right\|_{\ell_{j_1,\dots,j_k}^\infty \ell_m^1} \lesssim 1,$$

$$\left\|e^{\delta|j_1+\cdots+j_k-m|}D_{j_1}\dots D_{j_k}w(m)\right\|_{\ell_{m,j_2,\dots,j_k}^\infty \ell_{j_1}^1} \lesssim 1.$$

The rest of the argument follows as in other sections.

4.5.1. *Estimation of $\partial_{\bar{v}(k)}g_{F_1}^3(H) \circ X_{F_1}^s(v)$.* Since $g_{F_1}(\Lambda_2) = -H_4^{\text{nr}}$, it suffices to estimate

$$\sup_{s\in[0,1]} \left\|\frac{\partial}{\partial \bar{v}(k)}g_{F_1}^a(H_4) \circ X_{F_1}^s(v)\right\|_{\ell^p}, \quad a = 2,3.$$

When $a = 2$, we estimate this expression rather than the one containing H_4^{nr} (as we should have) because it simplifies the notation and still implies the estimate for the required expression. Note that

$$
\left\| \frac{\partial}{\partial \bar{v}(k)} g_{F_1}^a(H_4) \circ X_{F_1}^s(v) \right\|_{\ell_k^1} \leq \left\| \sum_i \frac{\partial g_{F_1}^a(H_4)}{\partial \bar{w}(i)} \frac{\partial \bar{w}(i)}{\partial \bar{v}(k)} \right\|_{\ell_k^1} + \|c.c.\|_{\ell_k^1}
$$

$$
\leq \left\| \frac{\partial g_{F_1}^a(H_4)}{\partial \bar{w}(i)} \right\|_{\ell_i^1} \left\| D_k \bar{w}(i) \right\|_{\ell_i^\infty \ell_k^1} + \|c.c.\|_{\ell_k^1}
$$

$$
\lesssim \varepsilon^{\frac{1}{2}-},
$$

for $a \geq 2$ by (28) and the estimates we obtained in Subsection 4.4. Similarly, using (29), we obtain

$$
\left\| \frac{\partial}{\partial \bar{v}(k)} g_{F_1}^a(H_4) \circ X_{F_1}^s(v) \right\|_{\ell_k^\infty} \lesssim \varepsilon^{\frac{3}{2}-}, \quad \text{for } a \geq 2.
$$

The estimates for $\partial_{\bar{v}(k)} \{ g_{F_1}^3(H) \circ X_{F_1}^s, F_2 \}$ and $\partial_{\bar{v}(k)} g_{F_2}^2(H \circ X_{F_1}^1) \circ X_{F_2}^s$ are similar. The only difference is that we also require the higher order derivative estimates of $w = X_{F_1}^s(v)$ listed above. We omit the details.

Appendix A. Nonlinear fiber optics application

One of the most important applications of NLS concerns light-wave communication systems, where optical pulses in a retarded time frame evolve according to the one dimensional NLS

$$
(30) \qquad\qquad iA_z + Sd(z)A_{\tau\tau} + g(z)|A|^2 A = 0.
$$

Here z is the rescaled distance, τ is the rescaled retarded time, A is the amplitude of the optical wave envelope, $d(z)$ is the group velocity dispersion, which is usually piecewise constant, and S is the dispersion strength parameter. Finally, $g(z) > 0$ is the nonlinear coefficient which accounts for the losses and amplifications. For the derivation of NLS from Maxwell's equations, one can consult many references, *e.g.* [9]. It is a standard assumption that $d(z)$ and $g(z)$ are periodic.

In general, in light-wave communication, the information is transmitted with localized pulses (with Gaussian or exponential tails) in allocated time slots. The presence of pulse corresponds to "1" and the absence of pulse corresponds to "0" in binary format. Naturally, it is preferable that the incoming waveform would appear undistorted at the end of the transmission line. It can be achieved by optimizing an individual pulse, so it would propagate without distortion, and sending such pulses together, keeping them sufficiently far apart (*i.e.* taking time slots sufficiently large), so they would not interact. Such regime is usually called "soliton regime" in the optical communication literature, where the word "soliton" does not usually mean that the equation is integrable. The pulses could be, for example, dispersion managed solitons, which are approximately periodic localized solutions of the above equation. In other words, the main feature of the soliton regime

is that the pulses do not interact (or rather pulse to pulse interaction is weak compared to the pulse self-interaction) during the propagation through the transmission line.

An alternative regime (often called the quasi-linear regime) has been found when the pulses strongly overlap during the transmission, see *e.g.* the survey paper [7]. Surprisingly, it was observed that up to a linear transformation of the transmitted waveform, the pulses appeared undistorted. Note that even though the pulses spread over many time slots, the average optical energy (L^2 norm square) per bit does not change and therefore nonlinear effects remain strong. It is usually implicitly assumed in the engineering literature that "nonlinearity gets averaged out" due to the high frequency of the initial data.

In this article, we rigorously explain the quasi-linear phenomenon for a model problem when $d(z)$ and $g(z)$ are constant and all bits are occupied by $1's$, in the limit of vanishing pulse width. This case (of all identical 1's) leads to the formulation with periodic boundary conditions. Although, this is a special case, we hope that our proof can be extended to the more general case: pseudo-random sequence of $1's$ and $0's$. Note that constant $d(z)$ and $g(z)$ assumption is not restrictive since if the evolution is quasi-linear on each interval where $d(z), g(z)$ are constant, then the evolution is quasi-linear on their union.

There has been previous work on quasi-linear regime. In [15], the limit of the short pulse width for dispersion managed NLS on the real line is considered. An effective evolution equation was derived which turned out to be integrable and weakly nonlinear. The equation was later improved in [1]. On the real line the energy disperses to infinity and therefore nonlinearity becomes small. This leaves an open question: *what will happen if the energy does not disperse to infinity or in other words, there is an infinite bit stream.* The problem considered in this paper models this situation: nonlinearity remains strong which is due to the periodic boundary conditions.

Finally, we note that on $\mathbb{R}$, the dispersion strength S and pulse width ε can be combined into a single effective parameter S/ε^2 by scaling τ. This implies that the limits $S \to \infty$ and $\varepsilon \to 0$ are equivalent. This is not the case in our model since the characteristic τ-scale, bit size, is already present. Therefore, the two parameter problem in S, ε should be considered. However, the limit $S \to \infty$ is insufficient to achieve quasi-linear evolution and must be supplemented with $\varepsilon \to 0$. On the other hand, the limit $\varepsilon \to 0$ does produce quasi-linear evolution with S being fixed but arbitrary. This motivated us to consider only this case. We put $S = 1$ in order not to obscure the exposition.

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