

# GLOBAL SMOOTHING FOR THE PERIODIC KdV EVOLUTION

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ABSTRACT. The Korteweg-de Vries (KdV) equation with periodic boundary conditions is considered. It is shown that for  $H^s$  initial data,  $s > -1/2$ , and for any  $s_1 < \min(3s+1, s+1)$ , the difference of the nonlinear and linear evolutions is in  $H^{s_1}$  for all times, with at most polynomially growing  $H^{s_1}$  norm. The result also extends to KdV with a smooth, mean zero, time-dependent potential in the case  $s \geq 0$ . Our result and a theorem of Oskolkov for the Airy evolution imply that if one starts with continuous and bounded variation initial data then the solution of KdV (given by the  $L^2$  theory of Bourgain) is a continuous function of space and time. In addition, we demonstrate smoothing for the defocusing modified KdV equation on the torus for  $s > 1/2$ .

## 1. INTRODUCTION

In this paper we study the Korteweg de Vries (KdV) equation on the torus

$$(1) \quad \begin{cases} u_t + u_{xxx} + 2uu_x = 0, & x \in \mathbb{T}, \quad t \in \mathbb{R}, \\ u(x, 0) = g(x) \in H^s(\mathbb{T}), \end{cases}$$

as well as the perturbed version with a smooth, mean-zero space-time potential

$$(2) \quad \begin{cases} u_t + u_{xxx} + (u^2 + \lambda u)_x = 0, & x \in \mathbb{T}, \quad t \in \mathbb{R}, \\ u(x, 0) = g(x) \in H^s(\mathbb{T}). \end{cases}$$

For both equations, we prove that the difference of the nonlinear evolution with the Airy evolution is smoother than both the Airy and the nonlinear evolutions. This smoothing property is not apparent if one views the nonlinear evolution as a perturbation of the linear flow and apply standard Picard iteration techniques to absorb the nonlinear derivative term. The result will follow from a combination of the method of normal forms (through differentiation by parts) inspired by the result in [1] and the restricted norm method of

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Bourgain, [2]. As it is well-known, KdV is a completely integrable system with infinitely many conserved quantities. However, our method in this paper does not rely on the integrability structure of KdV, and thus it can be applied to other dispersive models. For example even if we perturb the KdV equation with a smooth potential and break the integrability structure, we are still able to maintain the same smoothing properties.

The history of the KdV equation is quite rich and the literature is extensive. In this introduction we summarize some recent developments that are most relevant to our result. Note that all results were proved in the subset of solutions that have mean zero. This assumption can easily be dropped as we explain later in the paper since smooth solutions of KdV satisfy the momentum conservation:

$$\int_{-\pi}^{\pi} u(x, t) dx = \int_{-\pi}^{\pi} u(x, 0) dx.$$

To state some known results we start with a definition:

**Definition 1.1.** *Let  $X$  be a Banach space. We say that the equation (1) or (2) is locally well-posed in  $H^s(\mathbb{T})$ , if for a given initial data  $g \in H^s(\mathbb{T})$ , there exists  $T > 0$  and a unique solution  $u \in X \cap C_t^0 H_x^s([0, T] \times \mathbb{T})$ . We also demand that there is continuity with respect to the initial data in the appropriate topology. If  $T$  can be taken to be arbitrarily large then we say that the problem is globally well-posed.*

Well-posedness for nonsmooth data was first derived by Bourgain, [2]. He proved that the KdV equation is locally well-posed in  $L^2(\mathbb{T})$ . The existence of local-in-time solutions for KdV was investigated in a class of function spaces  $X$  and  $Y$  that satisfy inequalities of the form

$$\|\partial_x(u^2)\|_Y \leq C \|u\|_X^2.$$

The definition of the spaces adopted to the KdV equation will be given later but the reader should keep in mind that these spaces will incorporate the dispersive character of the equation. The space-time Fourier transform of the linear solution is supported on the characteristic surface,  $\tau = k^3$ . Bourgain observed that if one first localizes in time, then the Fourier transform of the nonlinear solution still concentrates near the characteristic surface due to the dispersive smoothing effect of  $\partial_t + \partial_x^3$ . Later, in [17], Kenig, Ponce, Vega

extended the local theory to  $H^s$ ,  $s \geq -\frac{1}{2}$ , and proved that the estimate above fails for any  $s < -\frac{1}{2}$ .

For  $L^2$  data these results can be extended to the case of arbitrary smooth perturbations, [22]. Due to the energy conservation:

$$\int_{-\pi}^{\pi} u^2(x, t) dx = \int_{-\pi}^{\pi} u^2(x, 0) dx$$

the KdV evolution is globally well-posed in  $L^2(\mathbb{T})$  and the solution is in  $C(\mathbb{R}; L^2(\mathbb{T}))$ . Colliander, Keel, Staffilani, Takaoka, Tao, [8], subsequently showed that KdV is globally well-posed in  $H^s(\mathbb{T})$  for any  $s \geq -\frac{1}{2}$ . To extend the local solutions globally in time they used the “I-method”, developing a theory of almost conserved quantities starting with the energy. Although the initial data have infinite energy they showed that a smoothed out version of the solution cannot increase much in energy going from one local-in-time interval to another. The iteration of this method leads to polynomial in time bounds for the rough Sobolev norm  $H^s$ , establishing the result. In [16], Kappeler and Topalov extended the latter result using the integrability properties of the equation and proved that KdV admits global solutions in  $H^s(\mathbb{T})$  for any  $s \geq -1$ .

Recently, in [1], Babin, Ilyin, and Titi gave a new proof of the  $L^2$  theorem of Bourgain using normal form methods. Similar ideas were developed by Shatah [21]. This method was also used in [12] and [13] to study near-linear behavior of KdV. In [1], the authors wrote the equation on the Fourier side and used differentiation by parts taking advantage of the large denominators that appear due to the dispersion. The method then proceeds by obtaining estimates only on  $L_t^\infty H_x^s$  norms. The clear advantage of their method apart from the simplicity is that they can obtain (a property that was observed by Kwon and Oh in [19]) unconditional uniqueness for KdV in  $C_t^0 L_x^2$ . In light of the theorem of Christ, in [6], this result appears to be sharp. Note that the method in [1] does not work in negative indexed Sobolev spaces. This is because a notion of distributional solutions (without using any auxiliary function space) is not well defined below  $L^2$  due to the quadratic nonlinearity in KdV. We will also use the differentiation by parts method in our proof. After one differentiation by parts we apply the restricted norm method of Bourgain which enables us to obtain a smoother solution starting with rough initial data for the full problem.

### 1.1. Main Results.

**Theorem 1.2.** *Fix  $s > -1/2$  and  $s_1 < \min(3s+1, s+1)$ . Consider the real valued solution of KdV (1) on  $\mathbb{T} \times \mathbb{R}$  with initial data  $u(x, 0) = g(x) \in H^s$ . Assume that we have a growth bound  $\|u(t)\|_{H^s} \leq C(\|g\|_{H^s})(1+|t|)^{\alpha(s)}$  for some  $\alpha(s)$ . Then  $u(t) - e^{tL}g \in C_t^0 H_x^{s_1}$  and*

$$\|u(t) - e^{tL}g\|_{H^{s_1}} \leq C(s, s_1, \|g\|_{H^s})(1+|t|)^{1+\beta(s)\alpha(s)}.$$

Here  $\beta(s) = \frac{15+6s}{3+2s} + \varepsilon$  and  $L = -\partial_x^3 + \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} g\right) \partial_x$ .

**Remark 1.** Note that for any  $s > -1/3$  and  $g \in H^s$ , Theorem 1.2 and Theorem 2.2 below imply that

$$u(t) - e^{tL}g \in L^2(\mathbb{T})$$

for all times. We thus obtain a periodic analogue of Bourgain's theorem in [4]. In this paper he developed a method of high-low frequency decomposition of the initial data to obtain global solutions with infinite energy for the NLS equation on  $\mathbb{R}^2$ . In addition he proved that the nonlinear term is smoother than the initial data.

**Remark 2.** We note that, within the method of differentiation by parts, the range for  $s_1$  when  $s \leq 0$  in the theorem seems to be optimal up to the endpoint. This is because of the resonant term

$$\left| \frac{u_k |u_k|^2}{k} \right| = (|u_k| |k|^s)^3 |k|^{-3s-1}$$

appearing after the first differentiation by parts. For general  $H^s$  data, this term can not be in  $H^{s_1}$  if  $s_1 > 3s+1$ . This also implies that for  $s = -1/2$  there is no smoothing within the tools that we use.

**Remark 3.** In the appendix, we demonstrate an analogous statement for the modified KdV equation on the torus.

**Theorem 1.3.** *Fix  $s \geq 0$  and  $s_1 < s+1$ . Consider (2) where  $\lambda \in C^\infty(\mathbb{T} \times \mathbb{R})$  is a mean zero real-valued potential with bounded derivatives and initial data  $u(x, 0) = g(x) \in H^s$ . Assume that we have a growth bound  $\|u(t)\|_{H^s} \leq C(\|g\|_{H^s})T(t)$  for some nondecreasing function  $T$  on  $[0, \infty)$ . Then  $u(t) - e^{tL}g \in C_t^0 H_x^{s_1}$  and*

$$\|u(t) - e^{tL}g\|_{H^{s_1}} \leq C(s, s_1, \|g\|_{H^s})(1+|t|)T(t)^9.$$

Here  $L = -\partial_x^3 + \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} g\right) \partial_x$ .

**Remark 4.** We should note that the only reason that we are restricted to the range  $s \geq 0$  in this theorem is the lack of global solutions at regularity levels below  $L^2$ .

**Remark 5.** For  $L^2$  initial data  $g$ , Theorem 1.3 implies that  $u - e^{tL}g \in C_t^0 H_x^{1-}$ , and hence is a continuous function of  $x$  and  $t$ .

Using this remark and the following theorem of Oskolkov, [20], we obtain Corollary 1.5 below. We also note that using our theorem it is likely that other known properties of the Airy evolution could be extended to the KdV evolution.

**Theorem 1.4.** [20] *Let  $L$  be as in the previous theorems and assume that  $g$  is of bounded variation, then  $e^{tL}g$  is a continuous function of  $x$  if  $t/2\pi$  is an irrational number. For rational values of  $t/2\pi$ , it is a bounded function with at most countably many discontinuities. Moreover, if  $g$  is also continuous then  $e^{tL}g \in C_t^0 C_x^0$ .*

**Corollary 1.5.** *Let  $u$  be the real valued solution of (2) with initial data  $g \in BV \subset L^2$ . Then,  $u$  is a continuous function of  $x$  if  $t/2\pi$  is an irrational number. For rational values of  $t/2\pi$ , it is a bounded function with at most countably many discontinuities. Moreover, if  $g$  is also continuous then  $u \in C_t^0 C_x^0$ .*

**Remark 6.** Note that if  $g \in H^s$  for some  $s > 1/2$ , then  $u \in C_t^0 H_x^s \subset C_t^0 C_x^0$  since KdV is globally well-posed. Oskolkov's theorem and smoothing (Theorem 1.2) allow us to have the same conclusion with initial data  $g \in BV \cap C^0 \subset \bigcap_{s < 1/2} H^s$ .

The smoothing properties that we describe for KdV are in a way unexpected. If we denote by  $X^s$  and  $Y^s$  the local theory spaces of the introduction at a regularity level that accommodates  $s$  derivatives, a smoothing estimate for the KdV will read as

$$\|\partial_x(u^2)\|_{Y^{s_1}} \leq C\|u\|_{X^s}^2$$

with  $s_1 > s$ . Unfortunately this estimate fails both on the torus and on the real line, [11].

There is no smoothing bound (as in Theorem 1.2) for KdV on the real line. In [11], the authors studied the existence of global-in-time solutions evolving from rougher than  $L^2$  data. In the absence of a smoothing estimate they managed instead to prove a modified version in the case that the functions had spatial Fourier transform supported away from zero. The failure of the estimate for wave interactions with low frequencies forced them

to consider data in a Sobolev type space which suppresses low frequencies. This result didn't improve Bourgain's result since the space they considered fails to contain  $L^2$ . On the other hand, Colliander, Delort, Kenig and Staffilani in [7], obtained bilinear smoothing estimates of the above form for NLS on  $\mathbb{R}^2$  but with special nonlinearities. They used these estimates to obtain local well-posedness results for certain bilinear Schrödinger equations and establish polynomial in time bounds for the higher order Sobolev norms of nonintegrable NLS equations.

For the periodic problem the first bilinear smoothing estimate that we know of is due to Kenig, Ponce and Vega, in [18] (also see [24]), for the 1d NLS on the torus and with the nonlinearity of the form  $u^2$ . In both cases ([18], [24]) the nonlinearities are not physical (the equations do not conserve the  $L^2$  norm), and they are useful in conjunction with other tools. We on the other hand prove a smoothing trilinear estimate for the full KdV nonlinearity after the normal form reduction. It is easy to see that with the  $uu_x$  nonlinearity the bilinear smoothing fails on the torus. In [5], Christ obtained a (local-in-time) smoothing estimate in the  $\mathcal{F}^{\ell^p} \rightarrow \mathcal{F}^{\ell^q}$  setting for 1d cubic NLS. Here  $\mathcal{F}^{\ell^p}$  is the space of functions whose Fourier series is in  $\ell^p$ . Thus, although the dynamics of periodic dispersive equations is a well studied subject, it appears that our result is the first smoothing estimate in Sobolev spaces for a physical partial differential equation<sup>1</sup>.

We say a few words about the method of the proof. Following the argument in [1] and using a normal form reduction we first rewrite the equation in an equivalent but easier form to deal with. In this particular form the derivative in the nonlinearity is eliminated. The penalty one pays after such a reduction is to increase the order of the nonlinearity (in KdV from quadratic to cubic) and to obtain resonant terms. Due to the absence of the zero Fourier modes (momentum conservation) the bilinear nonlinearity has no resonant terms. In addition we avoid the low frequency interactions that the authors in [11] faced. To estimate the new tri-linear term we now decompose the nonlinearity into resonant and nonresonant terms. It should be noted that in the resonant terms the waves interact with no oscillation and hence they are always "the enemy". Any method to estimate nonlinear

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<sup>1</sup>A smoothing result appeared in [15] during the review process of our paper. Although the result is similar to ours in spirit the basic difference is that, in [15], the KdV solution is approximated by a linear evolution which depends on the initial data in a nonlinear way.

interactions of dispersive equations reaches its limits exactly due to these terms (see Remark 2 above). But it turns out that the nonsmooth resonant terms of the KdV cancel out and the gain of the derivative is more than enough to compensate for the remaining nonlinear terms. What is more striking is the fact that the nonsmooth resonant terms cancel out even for the KdV equation with a potential. For the nonresonant terms, we apply the restricted norm method of Bourgain to the reduced nonlinearity to prove the smoothing. We note that a similar combination of integration by parts and the restricted norm method was used by Takaoka and Tsutsumi in [23].

**1.2. Notation.** To avoid the use of multiple constants, we write  $A \lesssim B$  to denote that there is an absolute constant  $C$  such that  $A \leq CB$ . We also write  $A \approx B$  to denote both  $A \lesssim B$  and  $B \lesssim A$ . We define  $\langle \cdot \rangle = 1 + |\cdot|$ .

We define the Fourier sequence of a  $2\pi$ -periodic  $L^2$  function  $u$  as

$$u_k = \frac{1}{2\pi} \int_0^{2\pi} u(x) e^{-ikx} dx, \quad k \in \mathbb{Z}.$$

With this normalization we have

$$u(x) = \sum_k e^{ikx} u_k, \quad \text{and } (uv)_k = u_k * v_k = \sum_{m+n=k} u_n v_m.$$

As usual, for  $s < 0$ ,  $H^s$  is the completion of  $L^2$  under the norm

$$\|u\|_{H^s} = \|\widehat{u}(k) \langle k \rangle^s\|_{\ell^2}.$$

Note that for a mean-zero  $L^2$  function  $u$ ,  $\|u\|_{H^s} \approx \|\widehat{u}(k) |k|^s\|_{\ell^2}$ . For a sequence  $u_k$ , with  $u_0 = 0$ , we will use  $\|u\|_{H^s}$  notation to denote  $\|u_k |k|^s\|_{\ell^2}$ .

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## 2. SOME RESULTS ON KdV EVOLUTION

In this section we define the spaces that are commonly used in the theory of periodic KdV, and list the results that our proof relies on.

We define the  $X^{s,b}$  spaces for  $2\pi$ -periodic KdV via the norm

$$\|u\|_{X^{s,b}} = \|\langle k \rangle^s \langle \tau - k^3 \rangle^b \widehat{u}(k, \tau)\|_{L^2(dk)d\tau}.$$

We also define the restricted norm

$$\|u\|_{X_\delta^{s,b}} = \inf_{\tilde{u}=u \text{ on } [-\delta, \delta]} \|\tilde{u}\|_{X^{s,b}}.$$

The local well-posedness theory for periodic KdV was established in the space  $X^{s,1/2}$ . Unfortunately, this space fails to control the  $L_t^\infty H_x^s$  norm of the solution. To remedy this problem and ensure the continuity of KdV flow, the  $Y^s$  and  $Z^s$  spaces are defined in [14] and [8], based on the ideas of Bourgain, in [2], via the norms

$$\|u\|_{Y^s} = \|u\|_{X^{s,1/2}} + \|\langle k \rangle^s \widehat{u}(k, \tau)\|_{L^2(dk)L^1(d\tau)},$$

$$\|u\|_{Z^s} = \|u\|_{X^{s,-1/2}} + \left\| \frac{\langle k \rangle^s \widehat{u}(k, \tau)}{\langle \tau - k^3 \rangle} \right\|_{L^2(dk)L^1(d\tau)}.$$

One defines  $Y_\delta^s, Z_\delta^s$  accordingly. Note that if  $u \in Y^s$  then  $u \in L_t^\infty H_x^s$ .

We use the following theorems from [8]:

**Theorem 2.1.** [8] *For any  $s \geq -1/2$ , the initial value problem (1) is locally well-posed in  $H^s$ . In particular,  $\exists \delta \approx (1 + \|g\|_{H^s})^{-\frac{6}{3+2s}-\varepsilon}$  such that there exists a unique solution*

$$u \in C([-\delta, \delta]; H_x^s(\mathbb{T})) \cap Y_\delta^s$$

with

$$\|u\|_{X_\delta^{s,1/2}} \leq \|u\|_{Y_\delta^s} \leq C\|g\|_{H^s}.$$

**Theorem 2.2.** [8] *For any  $s \geq -1/2$ , the initial value problem (1) is globally well-posed in  $H^s$ . Moreover, for  $-1/2 \leq s < 0$ , there is a growth bound (with some  $\alpha(s) > 0$ )*

$$\|u\|_{H^s} \leq C(1 + |t|)^{\alpha(s)},$$

where  $C$  depends on  $\|g\|_{H^s}$ .

The next theorem concerns KdV with potential.



**Theorem 2.3.** *For any  $s \geq 0$ , the initial value problem (2) is locally well-posed in  $H^s$ . In particular,  $\exists \delta \approx (1 + \|g\|_{H^s})^{-6}$  such that there exists a unique solution*

$$u \in C([-\delta, \delta]; H_x^s(\mathbb{T})) \cap Y_\delta^s$$

with

$$\|u\|_{X_\delta^{s,1/2}} \leq \|u\|_{Y_\delta^s} \leq C\|g\|_{H^s}.$$

The proof of this theorem follows the arguments in [2] and [22] which won't be repeated here. Notice that  $\delta$  is different in this theorem since the equation is not scale invariant anymore. The dependence of  $\delta$  on the size of the initial data given in the theorem is due to the following reason. Bourgain, in [2], proved that

$$\left\| \psi_\delta(t) \int_0^t e^{L(t-r)} \partial_x(uv)(r) dr \right\|_{Y^s} \lesssim \delta^{1/6} \|u\|_{X_\delta^{s,1/2}} \|v\|_{X_\delta^{s,1/2}}.$$

Here  $\psi_\delta(t) := \psi(t/\delta)$ , where  $\psi \in C^\infty$  and supported on  $[-2, 2]$ , and  $\psi(t) = 1$  on  $[-1, 1]$ . The gain in  $\delta$  comes from the  $L_{t,x}^4$  Strichartz estimate that he obtained. Using this inequality one has to pick  $\delta \approx \|g\|_{H^s}^{-6}$  to close the contraction argument.

Now we show how one can extend the local solutions of (2) to global ones.

**Theorem 2.4.** *The initial value problem (2) is globally well-posed in  $L^2$ . In particular, any  $H^s$  norm for  $s \geq 0$  grows at most exponentially. Moreover, if  $\partial_x \lambda \in L_t^1 L_x^\infty$ , then the  $L^2$  norm remains bounded and any  $H^s$  norm for  $s > 0$  grows at most polynomially.*

*Proof.* Note that using the equation we obtain

$$\frac{d}{dt} \int_{\mathbb{T}} u^2 dx = 2 \int_{\mathbb{T}} uu_t dx = \int_{\mathbb{T}} u^2 \lambda_x dx.$$

Integrating in time we obtain

$$\|u(t)\|_2^2 \leq \|g\|_2^2 + \int_0^t \|\lambda_x(s)\|_{L^\infty} \|u(s)\|_2^2 ds.$$

Thus, by Gronwall's inequality, we obtain

$$\|u\|_2 \leq \|g\|_2 e^{\int_0^t \|\lambda_x(s)\|_{L^\infty} ds}.$$

Since  $\lambda_x$  is bounded

$$\|u\|_2 \leq \|g\|_2 e^{Ct}.$$

Moreover if  $\lambda_x \in L_t^1 L_x^\infty$ , then

$$\|u\|_2 \lesssim \|g\|_2.$$

In both cases, we can iterate the local solution and obtain a global-in-time solution evolving from an  $L^2$  data.

To obtain the growth bound for the higher order norms we use Theorem 1.3 repeatedly as follows. For  $s \in (0, 1)$ , using the theorem with  $s = 0$ ,  $s_1 = s$ , and  $T(t) = e^{Ct}$  or  $T(t) = C$  depending on the assumptions on  $\lambda_x$ , we obtain

$$\|u(t) - e^{tL}g\|_{H^s} \leq C_{\|g\|_2} \langle t \rangle T(t)^9,$$

which implies by the unitarity of the linear evolution that

$$\|u(t)\|_{H^s} \leq C_{\|g\|_2} \langle t \rangle T(t)^9.$$

One can continue iteratively with this process and reach any index  $s$ . □

We should note that, in [3], Bourgain studied the growth of higher order Sobolev norms for KdV and the nonlinear Schrödinger equation (NLS). The critical observation was that if one has a local estimate of the form

$$\|u(t)\|_{H^s} \leq \|u_0\|_{H^s} + \|u_0\|_{H^s}^{1-\delta}$$

for  $T = T(\|u_0\|_{H^1})$  then one can obtain

$$\|u(t)\|_{H^s} \leq C|t|^{\frac{1}{\delta}}.$$

This method was later refined by Staffilani in [22]. In particular, she proved that for the periodic KdV with a potential, the higher order Sobolev norms grow with a bound of the form  $t^{(s-1)+}$  for any large  $s$  assuming an a priori bound on the  $H^1$  norm. Staffilani's method can also be applied in the case of time-dependent potentials<sup>2</sup>, and for  $s > 1$  it gives a better polynomial growth bound than we obtained above, if one knows that the  $H^1$  norm remains bounded. Our result on the other hand gives some polynomial growth bounds for any  $s \geq 0$ .

A final ingredient in our proof is the following Strichartz estimate of Bourgain [2]:

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<sup>2</sup>Personal communication with G. Staffilani

**Proposition 2.5.** [2] *For any  $\varepsilon > 0$  and  $b > 1/2$ , we have*

$$\|\chi_{[-\delta, \delta]}(t)u\|_{L_{t,x}^6(\mathbb{R} \times \mathbb{T})} \leq C_{\varepsilon, b} \|u\|_{X_{\delta}^{\varepsilon, b}}.$$

### 3. PROOFS OF THEOREM 1.2 AND THEOREM 1.3

We will give the proofs of these theorems simultaneously. This implies that Theorem 1.3 would be valid also for  $s \in (-1/2, 0)$  if one knew the required well-posedness theory in this regularity level.

We will give the proof under the assumption that the initial data (and hence the solution at each time) is mean zero. To remove this assumption one changes the equation introducing two terms of the form  $(\frac{1}{\pi} \int_{-\pi}^{\pi} g)u_x$  and  $(\frac{1}{2\pi} \int_{-\pi}^{\pi} g)\lambda_x$ . The first term changes the linear operator from  $-\partial_x^3$  to  $L$  as it is stated in the theorem. We note that after this change the resonances and the multilinear estimates remain the same. The second term is in  $H^s$  for any  $s$ , and in the calculations below it will only go into the  $B$  operator defined below, which satisfies the same estimates.

Using the notation  $u(x, t) = \sum_k u_k(t)e^{ikx}$  and  $\lambda(x, t) = \sum_k \lambda_k(t)e^{ikx}$ , we write (2) on the Fourier side:

$$\partial_t u_k = ik^3 u_k - ik \sum_{k_1+k_2=k} (\lambda_{k_1} + u_{k_1})u_{k_2}, \quad u_k(0) = \widehat{g}(k),$$

Because of the mean zero assumption on  $u$  and  $\lambda$ , there are no zero harmonics in this equation. Using the transformations

$$\begin{aligned} u_k(t) &= v_k(t)e^{ik^3 t}, \\ \lambda_k(t) &= \Lambda_k(t)e^{ik^3 t}, \end{aligned}$$

and the identity

$$(k_1 + k_2)^3 - k_1^3 - k_2^3 = 3(k_1 + k_2)k_1 k_2,$$

the equation can be written in the form

$$(3) \quad \partial_t v_k = -ik \sum_{k_1+k_2=k} e^{-3ikk_1 k_2 t} (v_{k_1} + \Lambda_{k_1})v_{k_2}.$$

We start with the following proposition which follows from differentiation by parts. A similar proposition was proved in [13].

**Proposition 3.1.** *The system (3) can be written in the following form:*

$$(4) \quad \partial_t[v + B(\Lambda + v, v)]_k = \rho_k + B(\partial_t\Lambda, v)_k + R(\Lambda + 2v, \Lambda + v, v)_k,$$

where we define  $B(f, g)_0 = \rho_0 = R(f, g, h)_0 = 0$ , and for  $k \neq 0$ , we define

$$\begin{aligned} B(f, g)_k &= -\frac{1}{3} \sum_{k_1+k_2=k} \frac{e^{-3ikk_1k_2t} f_{k_1} g_{k_2}}{k_1 k_2} \\ \rho_k &= \frac{i}{3} \Lambda_k \sum_{|j| \neq |k|} \frac{\Lambda_j \bar{v}_j}{j} - \frac{i}{3} \frac{(\bar{\Lambda}_k + 2\bar{v}_k)(\Lambda_k + v_k)v_k}{k} \\ R(f, g, h)_k &= \frac{i}{3} \sum_{\substack{k_1+k_2+k_3=k \\ (k_1+k_2)(k_1+k_3)(k_2+k_3) \neq 0}} \frac{e^{-3it(k_1+k_2)(k_2+k_3)(k_3+k_1)}}{k_1} f_{k_1} g_{k_2} h_{k_3}. \end{aligned}$$

*Proof of Proposition 3.1.* Since  $e^{-3ikk_1k_2t} = \partial_t(\frac{i}{3kk_1k_2}e^{-3ikk_1k_2t})$ , using differentiation by parts we can rewrite (3) as

$$\partial_t v_k = \partial_t \left( \sum_{k_1+k_2=k} \frac{e^{-3ikk_1k_2t} (\Lambda_{k_1} + v_{k_1})v_{k_2}}{3k_1k_2} \right) - \sum_{k_1+k_2=k} \frac{e^{-3ikk_1k_2t}}{3k_1k_2} \partial_t [(\Lambda_{k_1} + v_{k_1})v_{k_2}].$$

Recalling the definition of  $B$ , we can rewrite this equation in the form:

$$(5) \quad \partial_t[v + B(\Lambda + v, v)]_k = B(\partial_t\Lambda, v)_k - \frac{1}{3} \sum_{k_1+k_2=k} \frac{e^{-3ikk_1k_2t}}{k_1k_2} [v_{k_2} \partial_t v_{k_1} + (\Lambda_{k_1} + v_{k_1}) \partial_t v_{k_2}].$$

Note that since  $v_0 = 0$ , in the sums above  $k_1$  and  $k_2$  are not zero. Using (3), we have

$$\begin{aligned} -\frac{1}{3} \sum_{k_1+k_2=k} \frac{e^{-3ikk_1k_2t}}{k_1k_2} v_{k_2} \partial_t v_{k_1} &= \frac{i}{3} \sum_{k=k_1+k_2} \frac{e^{-3ikk_1k_2t} v_{k_2}}{k_2} \sum_{\mu+\nu=k_1 \neq 0} e^{-3itk_1\mu\nu} (\Lambda_\mu + v_\mu)v_\nu \\ &= \frac{i}{3} \sum_{\substack{k=k_2+\mu+\nu \\ \mu+\nu \neq 0}} \frac{e^{-3it[kk_2(\mu+\nu)+\mu\nu(\mu+\nu)]}}{k_2} v_{k_2} (\Lambda_\mu + v_\mu)v_\nu. \end{aligned}$$

Using the identity

$$kk_2 + \mu\nu = (k_2 + \mu + \nu)k_2 + \mu\nu = (k_2 + \mu)(k_2 + \nu)$$

and by renaming the variables  $k_2 \rightarrow k_1$ ,  $\mu \rightarrow k_2$ ,  $\nu \rightarrow k_3$ , we have that

$$(6) \quad -\frac{1}{3} \sum_{k_1+k_2=k} \frac{e^{-3ikk_1k_2t}}{k_1k_2} v_{k_2} \partial_t v_{k_1}$$

$$= \frac{i}{3} \sum_{\substack{k_1+k_2+k_3=k \\ k_2+k_3 \neq 0}} \frac{e^{-3it(k_1+k_2)(k_2+k_3)(k_3+k_1)}}{k_1} v_{k_1} (\Lambda_{k_2} + v_{k_2}) v_{k_3}.$$

Similarly,

$$(7) \quad -\frac{1}{3} \sum_{k_1+k_2=k} \frac{e^{-3ikk_1k_2t}}{k_1k_2} (\Lambda_{k_1} + v_{k_1}) \partial_t v_{k_2} \\ = \frac{i}{3} \sum_{\substack{k_1+k_2+k_3=k \\ k_2+k_3 \neq 0}} \frac{e^{-3it(k_1+k_2)(k_2+k_3)(k_3+k_1)}}{k_1} (\Lambda_{k_1} + v_{k_1}) (\Lambda_{k_2} + v_{k_2}) v_{k_3}.$$

Combining (6) and (7), we can rewrite (5) as

$$\partial_t[v + B(\Lambda + v, v)]_k = B(\partial_t \Lambda, v)_k \\ + \frac{i}{3} \sum_{\substack{k_1+k_2+k_3=k \\ k_2+k_3 \neq 0}} \frac{e^{-3it(k_1+k_2)(k_2+k_3)(k_3+k_1)}}{k_1} (\Lambda_{k_1} + 2v_{k_1}) (\Lambda_{k_2} + v_{k_2}) v_{k_3}.$$

Note that the set on which the phase on the right hand side vanishes is the disjoint union of the following sets

$$S_1 = \{k_1 + k_2 = 0\} \cap \{k_3 + k_1 = 0\} \cap \{k_2 + k_3 \neq 0\} \Leftrightarrow \{k_1 = -k, k_2 = k, k_3 = k\},$$

$$S_2 = \{k_1 + k_2 = 0\} \cap \{k_3 + k_1 \neq 0\} \cap \{k_2 + k_3 \neq 0\} \Leftrightarrow \{k_1 = j, k_2 = -j, k_3 = k, |j| \neq |k|\},$$

$$S_3 = \{k_3 + k_1 = 0\} \cap \{k_1 + k_2 \neq 0\} \cap \{k_2 + k_3 \neq 0\} \Leftrightarrow \{k_1 = j, k_2 = k, k_3 = -j, |j| \neq |k|\}.$$

Thus, using the definition of  $R(u, v, w)$ , we have

$$\partial_t[v + B(\Lambda + v, v)]_k = B(\partial_t \Lambda, v)_k + R(\Lambda + 2v, \Lambda + v, v)_k \\ + \frac{i}{3} \sum_{\ell=1}^3 \sum_{S_\ell} \frac{(\Lambda_{k_1} + 2v_{k_1})(\Lambda_{k_2} + v_{k_2}) v_{k_3}}{k_1}.$$

The proposition follows if we show that the second line above is equal to  $\rho_k$ . Note that

$$(8) \quad \sum_{\ell=1}^3 \sum_{S_\ell} \frac{(\Lambda_{k_1} + 2v_{k_1})(\Lambda_{k_2} + v_{k_2}) v_{k_3}}{k_1} = -\frac{(\Lambda_{-k} + 2v_{-k})(\Lambda_k + v_k) v_k}{k} \\ + v_k \sum_{|j| \neq |k|} \frac{(\Lambda_j + 2v_j)(\Lambda_{-j} + v_{-j})}{j} + (\Lambda_k + v_k) \sum_{|j| \neq |k|} \frac{(\Lambda_j + 2v_j) v_{-j}}{j}$$

Note that using  $v_j = \overline{v_{-j}}$  and  $\Lambda_j = \overline{\Lambda_{-j}}$ , we can rewrite the second line above as

$$\begin{aligned} & v_k \sum_{|j| \neq |k|} \frac{|\Lambda_j + v_j|^2 + |v_j|^2 + v_j \overline{\Lambda_j}}{j} + (\Lambda_k + v_k) \sum_{|j| \neq |k|} \frac{(\Lambda_j \overline{v_j} + 2|v_j|^2)}{j} \\ &= v_k \sum_{|j| \neq |k|} \frac{v_j \overline{\Lambda_j}}{j} + (\Lambda_k + v_k) \sum_{|j| \neq |k|} \frac{\Lambda_j \overline{v_j}}{j} \\ &= 2v_k \sum_{|j| \neq |k|} \frac{\Re(v_j \overline{\Lambda_j})}{j} + \Lambda_k \sum_{|j| \neq |k|} \frac{\Lambda_j \overline{v_j}}{j}. \end{aligned}$$

The first equality follows from the symmetry relation  $j \leftrightarrow -j$ . By the same token, the first summand in the last line above vanishes since  $\Re(v_j \overline{\Lambda_j}) = \Re(\overline{v_j} \Lambda_j) = \Re(v_{-j} \overline{\Lambda_{-j}})$ . Using this in (8) we obtain

$$\begin{aligned} \sum_{\ell=1}^3 \sum_{S_\ell} \frac{(\Lambda_{k_1} + 2v_{k_1})(\Lambda_{k_2} + v_{k_2})v_{k_3}}{k_1} &= -\frac{(\Lambda_{-k} + 2v_{-k})(\Lambda_k + v_k)v_k}{k} + \Lambda_k \sum_{|j| \neq |k|} \frac{\Lambda_j \overline{v_j}}{j} \\ &= \frac{3}{i} \rho_k, \end{aligned}$$

which yields the assertion of the proposition.  $\square$

Integrating (4) from 0 to  $t$ , we obtain

$$\begin{aligned} v_k(t) - v_k(0) &= -B(\Lambda + v, v)_k(t) + B(\Lambda + v, v)_k(0) + \int_0^t B(\partial_r \Lambda, v)_k(r) dr \\ &\quad + \int_0^t \rho_k(r) dr + \int_0^t R(\Lambda + 2v, \Lambda + v, v)_k(r) dr. \end{aligned}$$

Transforming back to the  $u, \lambda$  variables, we have

$$\begin{aligned} (9) \quad u_k(t) - e^{ik^3 t} u_k(0) &= -\mathcal{B}(\lambda + u, u)_k(t) + e^{ik^3 t} \mathcal{B}(\lambda + u, u)_k(0) \\ &\quad + \int_0^t e^{ik^3(t-r)} \mathcal{B}(e^{rL} \partial_r(e^{-rL} \lambda), u)_k(r) dr + \int_0^t e^{ik^3(t-r)} \tilde{\rho}_k(r) dr \\ &\quad + \int_0^t e^{ik^3(t-r)} \mathcal{R}(\lambda + 2u, \lambda + u, u)_k(r) dr, \end{aligned}$$

where

$$\begin{aligned} \mathcal{B}(f, g)_k &= -\frac{1}{3} \sum_{k_1+k_2=k} \frac{f_{k_1} g_{k_2}}{k_1 k_2}, \\ \tilde{\rho}_k &= \frac{i}{3} \lambda_k \sum_{|j| \neq |k|} \frac{\lambda_j \overline{u_j}}{j} - \frac{i}{3} \frac{(\overline{\lambda_k} + 2\overline{u_k})(\lambda_k + u_k) u_k}{k} \end{aligned}$$

$$\mathcal{R}(f, g, h)_k = \frac{i}{3} \sum_{\substack{k_1+k_2+k_3=k \\ (k_2+k_3)(k_1+k_2)(k_1+k_3) \neq 0}} \frac{f_{k_1} g_{k_2} h_{k_3}}{k_1}.$$

**Lemma 3.2.** *For  $s > -1/2$  and  $s_1 \leq s + 1$ , we have*

$$\|\mathcal{B}(u, v)\|_{H^{s_1}} \lesssim \|u\|_{H^s} \|v\|_{H^s}.$$

*For  $s > -1/2$  and  $s \leq s_1 < 3s + 1$ , we have*

$$\|\tilde{\rho}\|_{H^{s_1}} \lesssim \|u\|_{H^s} (\|\lambda\|_{H^{s_1}}^2 + \|u\|_{H^s}^2).$$

*Proof.* By symmetry we can assume in the estimate for  $\mathcal{B}(u, v)$  that  $|k_1| \geq |k_2|$ . Thus, for  $s_1 < s + 1$  and  $s > -1/2$ , we have

$$\begin{aligned} \|\mathcal{B}(u, v)\|_{H^{s_1}} &\lesssim \left\| \sum_{k_1+k_2=k, |k_1| \geq |k_2|} \frac{|u_{k_1} v_{k_2}|}{|k_1 k_2|} \right\|_{H^{s_1}} \\ &\lesssim \left\| \sum_{k_1+k_2=k, |k_1| \geq |k_2|} \frac{|k_1|^s |u_{k_1}| |k_2|^s |v_{k_2}| |k|^{s_1-s-1}}{|k_2|^{s+1}} \right\|_2 \\ &\lesssim \left\| \sum_{k_1+k_2=k, |k_1| \geq |k_2|} \frac{|k_1|^s |u_{k_1}| |k_2|^s |v_{k_2}|}{|k_2|^{s+1}} \right\|_2 \\ &\lesssim \left\| \frac{|k|^s v_k}{|k|^{1+s}} \right\|_{\ell^1} \left\| |k|^s u_k \right\|_2 \lesssim \left\| |k|^s v_k \right\|_2 \left\| |k|^{-1-s} \right\|_2 \|u\|_{H^s} \lesssim \|u\|_{H^s} \|v\|_{H^s}. \end{aligned}$$

In the last line we used Young's inequality and Cauchy-Schwarz.

Now note that for  $s_1 < 3s + 1$ ,

$$\left\| \frac{u_k v_k w_k}{k} \right\|_{H^{s_1}} = \left\| |u_k v_k w_k| |k|^{3s} |k|^{s_1-3s-1} \right\|_2 \lesssim \left\| |u_k v_k w_k| |k|^{3s} \right\|_2 \lesssim \|u\|_{H^s} \|v\|_{H^s} \|w\|_{H^s}.$$

Also note that for any  $-1/2 \leq s$ ,

$$\begin{aligned} \left\| \lambda_k \sum \frac{|\lambda_j| |v_j|}{|j|} \right\|_{H^{s_1}} &\leq \|\lambda\|_{H^{s_1}} \sum \frac{|\lambda_j|}{|j|^{1+s}} |u_j| |j|^s \leq \|\lambda\|_{H^{s_1}} \|\lambda\|_{H^{-s-1}} \|u\|_{H^s} \\ &\leq \|\lambda\|_{H^{s_1}}^2 \|u\|_{H^s}. \end{aligned}$$

The last two estimates imply the bound for  $\tilde{\rho}$ .  $\square$

Using the estimates in Lemma 3.2 in the equation (9), we obtain (for  $s > -1/2$  and  $s_1 < \min(3s + 1, s + 1)$ )

$$\|u(t) - e^{tL} g\|_{H^{s_1}} \lesssim \|\lambda(t) + u(t)\|_{H^s} \|u(t)\|_{H^s} + \|\lambda(0) + g\|_{H^s} \|g\|_{H^s}$$

$$\begin{aligned}
& + \int_0^t \|e^{rL} \partial_r e^{-rL} \lambda(r)\|_{H^s} \|u(r)\|_{H^s} dr + \int_0^t \|u(r)\|_{H^s} (\|\lambda(r)\|_{H^{s_1}}^2 + \|u(r)\|_{H^s}^2) dr \\
& + \left\| \int_0^t e^{ik^3(t-r)} \mathcal{R}(\lambda + 2u, \lambda + u, u)_k(r) dr \right\|_{H^{s_1}} \\
(10) \quad & \lesssim \|u(t)\|_{H^s} + \|u(t)\|_{H^s}^2 + \|g\|_{H^s} + \|g\|_{H^s}^2 + \int_0^t (\|u\|_{H^s} + \|u\|_{H^s}^3) dr \\
& + \left\| \int_0^t e^{L(t-r)} \mathcal{R}(\lambda + 2u, \lambda + u, u)(r) dr \right\|_{H^{s_1}},
\end{aligned}$$

where the implicit constant in the second inequality depends on  $\lambda \in C^\infty$  and

$$\mathcal{R}(u, v, w)(r, x) = \sum_{k \neq 0} \mathcal{R}(u, v, w)_k(r) e^{ikx}.$$

Since our nonlinearity after differentiation by parts is not  $uu_x$  anymore, we will be able to avoid the  $Y^{s_1}$  and  $Z^{s_1}$  spaces. Instead we will use the embedding  $X^{s_1, b} \subset L_t^\infty H_x^{s_1}$  for  $b > 1/2$  and the following lemma from [14]. Let  $\psi_\delta(t) := \psi(t/\delta)$ , where  $\psi \in C^\infty$  and supported on  $[-2, 2]$ , and  $\psi(t) = 1$  on  $[-1, 1]$ .

**Lemma 3.3.** *For  $-\frac{1}{2} < b' \leq 0 \leq b \leq b' + 1$ , we have*

$$(11) \quad \left\| \psi_\delta(t) \int_0^t e^{L(t-r)} F(r) dr \right\|_{X^{s, b}} \lesssim \delta^{1-b+b'} \|F\|_{X_\delta^{s, b'}}.$$

For  $t \in [-\delta/2, \delta/2]$ , where  $\delta$  is as in Theorem 2.1 or Theorem 2.3, and  $b > 1/2$ ,

$$\begin{aligned}
(12) \quad & \left\| \int_0^t e^{L(t-r)} \mathcal{R}(\lambda + 2u, \lambda + u, u)(r) dr \right\|_{H^{s_1}} \\
& \leq \left\| \psi_\delta(t) \int_0^t e^{L(t-r)} \mathcal{R}(\lambda + 2u, \lambda + u, u)(r) dr \right\|_{L_t^\infty H_x^{s_1}} \\
& \lesssim \left\| \psi_\delta(t) \int_0^t e^{L(t-r)} \mathcal{R}(\lambda + 2u, \lambda + u, u)(r) dr \right\|_{X^{s_1, b}} \\
& \lesssim \delta^{\varepsilon/2} \|\mathcal{R}(\lambda + 2u, \lambda + u, u)\|_{X_\delta^{s_1, -\frac{1}{2} + \varepsilon}},
\end{aligned}$$

for sufficiently small  $\varepsilon > 0$ .

**Proposition 3.4.** *For  $s > -1/2$ ,  $s_1 < \min(s + 1, 3s + 1)$ , and  $\varepsilon > 0$  sufficiently small, we have*

$$\|\mathcal{R}(u, v, w)\|_{X_\delta^{s_1, -\frac{1}{2} + \varepsilon}} \leq C \|u\|_{X_\delta^{s, 1/2}} \|v\|_{X_\delta^{s, 1/2}} \|w\|_{X_\delta^{s, 1/2}}.$$



We will prove this proposition in Section 4. Using (12) and the proposition above in (10), we see that for  $t \in [-\delta/2, \delta/2]$ , we have (with implicit constant depending on  $\lambda$ )

$$\begin{aligned} \|u(t) - e^{tL}g\|_{H^{s_1}} &\lesssim \|u(t)\|_{H^s} + \|u(t)\|_{H^s}^2 + \|g\|_{H^s} + \|g\|_{H^s}^2 + \int_0^t (\|u\|_{H^s} + \|u\|_{H^s}^3) dr \\ &\quad + \|u\|_{X_\delta^{s,1/2}} + \|u\|_{X_\delta^{s,1/2}}^3. \end{aligned}$$

In the rest of the proof the implicit constants also depend on  $\|g\|_{H^s}$ . Fix  $t$  large. For  $r \leq t$ , we have the bound

$$\|u(r)\|_{H^s} \lesssim T(r) \leq T(t).$$

We assume  $T(t) \geq 1$  without loss of generality. Thus, by the local theory, with  $\delta \approx T(t)^{-6}$ , we have

$$\|u(j\delta) - e^{\delta L}u((j-1)\delta)\|_{H^{s_1}} \lesssim T(t)^3,$$

for any  $j \leq t/\delta \approx tT(t)^6$ . Here we used the local theory bound

$$\|u\|_{X_{[(j-1)\delta, j\delta]}^{s,1/2}} \lesssim \|u((j-1)\delta)\|_{H^s} \lesssim T(t).$$

Using this we obtain (with  $J = t/\delta \approx tT(t)^6$ )

$$\begin{aligned} \|u(J\delta) - e^{J\delta L}u(0)\|_{H^{s_1}} &\leq \sum_{j=1}^J \|e^{(J-j)\delta L}u(j\delta) - e^{(J-j+1)\delta L}u((j-1)\delta)\|_{H^{s_1}} \\ &= \sum_{j=1}^J \|u(j\delta) - e^{\delta L}u((j-1)\delta)\|_{H^{s_1}} \lesssim JT(t)^3 \approx tT(t)^9. \end{aligned}$$

This completes the proof of the growth bound stated in Theorem 1.3.

In the case of KdV without potential the local theory bound gives  $\delta \approx T(t)^{-\frac{6}{3+2s}-\varepsilon}$  instead of the  $T(t)^{-6}$  power. Also taking into account that  $T(t) = \langle t \rangle^{\alpha(s)}$ , we obtain the growth bound stated in Theorem 1.2.

Now we will prove the continuity of  $N(t) := u(t) - e^{tL}g$  in  $H^{s_1}$ . Using (9), we obtain

$$\begin{aligned} (13) \quad N(t) - N(\tau) &= \mathcal{B}(\lambda + u, u)_k(\tau) - \mathcal{B}(\lambda + u, u)_k(t) \\ &\quad + (e^{ik^3t} - e^{ik^3\tau})\mathcal{B}(\lambda + u, u)_k(0) \\ &\quad + \int_0^t e^{ik^3(t-r)} \mathcal{B}(e^{rL}\partial_r(e^{-rL}\lambda), u)_k(r) dr - \int_0^\tau e^{ik^3(\tau-r)} \mathcal{B}(e^{rL}\partial_r(e^{-rL}\lambda), u)_k(r) dr \\ &\quad + \int_0^t e^{ik^3(t-r)} \tilde{\rho}_k(r) dr - \int_0^\tau e^{ik^3(\tau-r)} \tilde{\rho}_k(r) dr \end{aligned}$$

$$+ \int_0^t e^{ik^3(t-r)} \mathcal{R}(\lambda + 2u, \lambda + u, u)_k(r) dr - \int_0^\tau e^{ik^3(\tau-r)} \mathcal{R}(\lambda + 2u, \lambda + u, u)_k(r) dr.$$

For fixed  $\tau$ , we will show that the  $H^{s_1}$  norm of each line in the formula above converges to zero as  $t \rightarrow \tau$ . We will assume that  $t > \tau$  without loss of generality. For the first line this follows by using the difference  $u(\tau) - u(t)$ , the continuity of the solution in  $H^s$ , and the a priori bounds in Lemma 3.2 for  $\mathcal{B}$ . For the second line, we use the inequality (for any given  $\varepsilon \in [0, 1]$ )

$$(14) \quad |e^{ik^3 t} - e^{ik^3 \tau}| \lesssim \min(1, |k|^3 |t - \tau|) \leq (|k|^3 |t - \tau|)^\varepsilon$$

and the a priori estimates for  $\mathcal{B}$  in  $H^{s_1+3\varepsilon}$  for sufficiently small  $\varepsilon > 0$ , to obtain a bound of the form  $|t - \tau|^\varepsilon$ . We now explain how to bound the fifth line. The third and the fourth lines can be treated similarly using  $H^s$  norms instead of  $X^{s,b}$  norms. We write the fifth line as

$$(15) \quad (e^{ik^3(t-\tau)} - 1) \int_0^\tau e^{ik^3(\tau-r)} \mathcal{R}(\lambda + 2u, \lambda + u, u)_k(r) dr$$

$$(16) \quad + \int_\tau^t e^{ik^3(t-r)} \mathcal{R}(\lambda + 2u, \lambda + u, u)_k(r) dr.$$

To estimate  $H^{s_1}$  norm of (15) we use (14) with sufficiently small  $\varepsilon > 0$  to obtain

$$\|(15)\|_{H^{s_1}} \lesssim |t - \tau|^\varepsilon \left\| \int_0^\tau e^{ik^3(\tau-r)} \mathcal{R}(\lambda + 2u, \lambda + u, u)_k(r) dr \right\|_{H^{s_1+3\varepsilon}}.$$

To estimate the norm we divide the integral into  $\tau/\delta$  pieces where  $\delta$  is given by the local  $X^{s,1/2}$  theory. Here  $\delta$  depends on  $\sup_{r \in [0, \tau]} \|u(r)\|_{H^s}$ , which is finite due to global well-posedness. Then, we use (12) and Proposition 3.4 (with  $s_1 + 3\varepsilon$ ) to estimate each integral. Finally, the bound for the  $H^{s_1}$  norm of (16) follows from the gain in  $\delta$  in (12) and Proposition 3.4.

#### 4. PROOF OF PROPOSITION 3.4

Recall that

$$\mathcal{R}(u, v, w)(r, x) = \sum_{k \neq 0} \mathcal{R}(u, v, w)_k(r) e^{ikx}.$$

We need to prove that

$$\|\mathcal{R}(u, v, w)\|_{X_\delta^{s_1, -1/2+\varepsilon}} \lesssim \|u\|_{X_\delta^{s, 1/2}} \|v\|_{X_\delta^{s, 1/2}} \|w\|_{X_\delta^{s, 1/2}}.$$

As usual this follows by considering the  $X^{s,b}$  norms instead of the restricted versions. By duality it suffices to prove that

$$(17) \quad \left| \sum_k \int_{\mathbb{R}} \widehat{\mathcal{R}}(k, \tau) \widehat{h}(-k, -\tau) d\tau \right| = \left| \int_{\mathbb{R} \times \mathbb{T}} \mathcal{R}(u, v, w) h \right| \\ \lesssim \|u\|_{X^{s,1/2}} \|v\|_{X^{s,1/2}} \|w\|_{X^{s,1/2}} \|h\|_{X^{-s,1,1/2-\varepsilon}}.$$

We note that

$$\widehat{\mathcal{R}}(k, \tau) = \frac{i}{3} \int_{\tau_1 + \tau_2 + \tau_3 = \tau} \sum_{\substack{k_1 + k_2 + k_3 = k \\ (k_2 + k_3)(k_1 + k_2)(k_1 + k_3) \neq 0}} \frac{\widehat{u}(k_1, \tau_1) \widehat{v}(k_2, \tau_2) \widehat{w}(k_3, \tau_3)}{k_1}.$$

Let

$$f_1(k, \tau) = |\widehat{u}(k, \tau)| |k|^s \langle \tau - k^3 \rangle^{1/2}, \\ f_2(k, \tau) = |\widehat{v}(k, \tau)| |k|^s \langle \tau - k^3 \rangle^{1/2}, \\ f_3(k, \tau) = |\widehat{w}(k, \tau)| |k|^s \langle \tau - k^3 \rangle^{1/2}, \\ f_4(k, \tau) = |\widehat{h}(k, \tau)| |k|^{-s_1} \langle \tau - k^3 \rangle^{1/2-\varepsilon}.$$

Note that (17) follows from

$$(18) \quad \sum_{\substack{k_1 + k_2 + k_3 + k_4 = 0 \\ (k_2 + k_3)(k_1 + k_2)(k_1 + k_3) \neq 0}} \int_{\tau_1 + \tau_2 + \tau_3 + \tau_4 = 0} \frac{|k_1 k_2 k_3|^{-s} |k_4|^{s_1} \prod_{i=1}^4 f_i(k_i, \tau_i)}{|k_1| \prod_{i=1}^4 \langle \tau_i - k_i^3 \rangle^{1/2-\varepsilon}} \lesssim \prod_{i=1}^4 \|f_i\|_2.$$

By Proposition 2.5, we have (for any  $\varepsilon > 0$ )

$$(19) \quad \left\| \left( \frac{f_i |k|^{-\varepsilon}}{\langle \tau - k^3 \rangle^{1/2+\varepsilon}} \right)^\vee \right\|_{L^6(\mathbb{R} \times \mathbb{T})} \lesssim \|f_i\|_2.$$

Using  $\tau_1 + \tau_2 + \tau_3 + \tau_4 = 0$  and  $k_1 + k_2 + k_3 + k_4 = 0$ , we have

$$\sum_{i=1}^4 \tau_i - k_i^3 = -k_1^3 - k_2^3 - k_3^3 - k_4^3 = 3(k_1 + k_2)(k_1 + k_3)(k_2 + k_3).$$

Therefore

$$\max_{i=1,2,3,4} \langle \tau_i - k_i^3 \rangle \gtrsim |k_1 + k_2| |k_1 + k_3| |k_2 + k_3|.$$

Since the inequality (18) is symmetric in  $f_i$ 's, it does not matter which of these terms is the maximum. Therefore without loss of generality we assume that

$$\langle \tau_1 - k_1^3 \rangle = \max_{i=1,2,3,4} \langle \tau_i - k_i^3 \rangle \gtrsim |k_1 + k_2| |k_1 + k_3| |k_2 + k_3|.$$

This implies that

$$(20) \quad \prod_{i=1}^4 \langle \tau_i - k_i^3 \rangle^{1/2-\varepsilon} \gtrsim (|k_1 + k_2||k_1 + k_3||k_2 + k_3|)^{1/2-7\varepsilon} \prod_{i=2}^4 \langle \tau_i - k_i^3 \rangle^{1/2+\varepsilon}.$$

Also note that (since all factors are nonzero and  $k_1 + k_2 + k_3 + k_4 = 0$ )

$$(21) \quad |k_1 + k_2||k_1 + k_3||k_2 + k_3| \gtrsim |k_i|, \quad i = 1, 2, 3, 4.$$

Now we will prove that for  $s > -1/2$ ,  $s_1 < \min(s + 1, 3s + 1)$  and for  $\varepsilon$  sufficiently small,

$$(22) \quad \frac{|k_1 k_2 k_3|^{-s} |k_4|^{s_1}}{|k_1|(|k_1 + k_2||k_1 + k_3||k_2 + k_3|)^{1/2-7\varepsilon}} \lesssim |k_1 k_2 k_3 k_4|^{-\varepsilon}.$$

By (21), this follows from

$$\frac{|k_1 k_2 k_3|^{-s} |k_4|^{s_1}}{|k_1|(|k_1 + k_2||k_1 + k_3||k_2 + k_3|)^{1/2-11\varepsilon}} \lesssim 1.$$

First consider the case  $s > -1/3$ ,  $s_1 < \min(3s + 1, s + 1)$ . Without loss of generality we can assume that  $s_1 \geq 0$ . Let  $M = \max(|k_1|, |k_2|, |k_3|)$ . Using  $|k_1||k_1 + k_2| \gtrsim |k_2|$  and  $|k_1||k_1 + k_3||k_3 + k_2| \gtrsim |k_2|$ , and by symmetry of  $k_2, k_3$ , we have

$$|k_1|(|k_1 + k_2||k_1 + k_3||k_2 + k_3|)^{1/2-11\varepsilon} \gtrsim M^{1-22\varepsilon}.$$

Thus

$$\frac{|k_1 k_2 k_3|^{-s} |k_4|^{s_1}}{|k_1|(|k_1 + k_2||k_1 + k_3||k_2 + k_3|)^{1/2-11\varepsilon}} \lesssim \frac{|k_1 k_2 k_3|^{-s} |k_4|^{s_1}}{M^{1-22\varepsilon}}.$$

Since  $|k_1 k_2 k_3|^{-s} \leq M^{-3s}$  for  $s < 0$  and  $|k_1 k_2 k_3|^{-s} \leq M^{-s}$  for  $s \geq 0$ , we have  $|k_1 k_2 k_3|^{-s} \lesssim M^{-\min(s, 3s)}$ . Using this, the inequality  $|k_4| \lesssim M$ , and  $0 \leq s_1 < \min(3s + 1, s + 1)$ , we bound the multiplier by

$$M^{-\min(s, 3s)} M^{s_1 - 1 + 22\varepsilon} \lesssim 1 \text{ for sufficiently small } \varepsilon.$$

Second consider the case  $-1/2 < s \leq -1/3$ ,  $s_1 < 3s + 1 = \min(3s + 1, s + 1) \leq 0$ . Using  $|k_4| = |k_1 + k_2 + k_3|$  and  $|k_1 + k_2 + k_3||k_2 + k_3| \gtrsim |k_1|$ , we have

$$\begin{aligned} & \frac{|k_1 k_2 k_3|^{-s} |k_4|^{s_1}}{|k_1|(|k_1 + k_2||k_1 + k_3||k_2 + k_3|)^{1/2-11\varepsilon}} \\ & \lesssim \frac{|k_2 k_3|^{-s}}{|k_1|^{1+s-s_1} (|k_1 + k_2||k_1 + k_3|)^{1/2-11\varepsilon} |k_2 + k_3|^{1/2+s_1-11\varepsilon}}. \end{aligned}$$

Now using  $|k_1||k_1 + k_i| \gtrsim |k_i|$ , we bound the multiplier by

$$\frac{|k_2 k_3|^{-s - \frac{1+s-s_1}{2}}}{(|k_1 + k_2||k_1 + k_3|)^{\frac{s_1-s}{2} - 11\varepsilon} |k_2 + k_3|^{1/2 + s_1 - 11\varepsilon}} \lesssim |k_2 k_3|^{\frac{s_1 - (3s+1)}{2}} \lesssim 1.$$

This finishes the proof of (22). Using (22) and (20) in (18) (and eliminating  $|k_1|^{-\varepsilon}$ ), we obtain

$$(18) \lesssim \sum_{k_1+k_2+k_3+k_4=0} \int_{\tau_1+\tau_2+\tau_3+\tau_4=0} \frac{|k_2 k_3 k_4|^{-\varepsilon} \prod_{i=1}^4 f_i(k_i, \tau_i)}{\prod_{i=2}^4 \langle \tau_i - k_i^3 \rangle^{1/2 + \varepsilon}}.$$

By Plancherel and the convolution structure, we can rewrite this as

$$\begin{aligned} \int_{\mathbb{R} \times \mathbb{T}} \widehat{f}_1(x, t) \prod_{i=2}^4 \left( \frac{f_i |k|^{-\varepsilon}}{\langle \tau - k^3 \rangle^{1/2 + \varepsilon}} \right)^\vee(x, t) &\leq \|f_1\|_{L^2(\mathbb{R} \times \mathbb{T})} \prod_{i=2}^4 \left\| \left( \frac{f_i |k|^{-\varepsilon}}{\langle \tau - k^3 \rangle^{1/2 + \varepsilon}} \right)^\vee \right\|_{L^6(\mathbb{R} \times \mathbb{T})} \\ &\lesssim \prod_{i=1}^4 \|f_i\|_2. \end{aligned}$$

In the last inequality we used (19).

## 5. APPENDIX: SMOOTHING FOR MODIFIED KDV

In this section we consider the defocusing modified KdV equation (mKdV) in the form

$$(23) \quad \begin{aligned} v_t + v_{xxx} &= 6v^2 v_x \\ v(x, 0) &= g(x) \in H^s(\mathbb{T}), \quad s > 1/2. \end{aligned}$$

The mKdV equation satisfies both mean and  $L^2$  conservation. Set  $\mu = \frac{1}{2\pi} \|v\|_{L^2}^2 = \frac{1}{2\pi} \|g\|_{L^2}^2$ .

Note that  $w(x, t) = v(x - 6\mu t, t)$  satisfies

$$(24) \quad \begin{aligned} w_t + w_{xxx} &= 6(w^2 - \frac{1}{2\pi} \|w\|_{L^2}^2) w_x \\ w(x, 0) &= g(x) \in H^s(\mathbb{T}), \quad s > 1/2. \end{aligned}$$

**Theorem 5.1.** *Fix  $s > 1/2$  and  $s_1 < \min(3s-1, s+1)$ . Consider the real valued solution of mKdV (23) on  $\mathbb{T} \times \mathbb{R}$  with mean-zero initial data  $v(x, 0) = g(x) \in H^s$ . Then  $v(t) - e^{tL} g \in C_t^0(\mathbb{R}; H_x^{s_1})$ , where  $L = -\partial_x^3 - \frac{3}{\pi} \|g\|_{L^2}^2 \partial_x$ .*

*Proof.* Let  $w$  be a solution of (24) with a mean-zero initial data  $g \in H^s(\mathbb{T})$ ,  $s > 1/2$ . We will use the Miura transform  $\mathbf{M}w := w_x + w^2 - \frac{1}{2\pi}\|w\|_{L^2}^2$ . Following [9], we note that if  $w$  solves (24) with a mean-zero initial data, then  $u = \mathbf{M}w$  solves the KdV equation

$$u_t + u_{xxx} = 6uu_x$$

with mean-zero initial data  $u(x, 0) = \mathbf{M}g(x)$ . The following theorem was proved in [9] (here  $H_0^s(\mathbb{T}) = \{u \in H^s(\mathbb{T}) : \widehat{u}(0) = 0\}$ ):

**Theorem 5.2.** [9] *Let  $s \geq 1/2$ . Then the map  $\mathbf{M}$  is a locally Lipschitz bijection from  $H_0^s(\mathbb{T})$  to  $H_0^{s-1}(\mathbb{T})$ , and the inverse map  $\mathbf{M}^{-1}$  is locally Lipschitz from  $H_0^{s-1}(\mathbb{T})$  to  $H_0^s(\mathbb{T})$ .*

This implies in particular that  $\mathbf{M}$  maps  $C_t^0(\mathbb{R}; H_0^s(\mathbb{T}))$  to  $C_t^0(\mathbb{R}; H_0^{s-1}(\mathbb{T}))$ , and  $\mathbf{M}^{-1}$  maps  $C_t^0(\mathbb{R}; H_0^{s-1}(\mathbb{T}))$  to  $C_t^0(\mathbb{R}; H_0^s(\mathbb{T}))$ . The theorem will follow from this and Theorem 1.2. Indeed, by Theorem 1.2,  $u = \mathbf{M}w$  satisfies

$$u(t) - e^{-\partial_x^3 t} \mathbf{M}g \in C_t^0(\mathbb{R}; H_0^\rho(\mathbb{T}))$$

for any  $\rho < \min(3s - 2, s)$ . This implies that

$$(25) \quad \mathbf{M}^{-1}u(t) - \mathbf{M}^{-1}e^{-\partial_x^3 t} \mathbf{M}g \in C_t^0(\mathbb{R}; H_0^{s_1}(\mathbb{T}))$$

for any  $s_1 < \min(3s - 1, s + 1)$ . In addition, since  $\partial_x$  and  $e^{-\partial_x^3 t}$  commutes, we have

$$\begin{aligned} e^{-\partial_x^3 t} \mathbf{M}g - \mathbf{M}e^{-\partial_x^3 t} g &= (e^{-\partial_x^3 t} g)^2 - e^{-\partial_x^3 t} (g^2) - \frac{1}{2\pi} \|e^{-\partial_x^3 t} g\|_{L^2}^2 + \frac{1}{2\pi} \|g\|_{L^2}^2 \\ &= (e^{-\partial_x^3 t} g)^2 - e^{-\partial_x^3 t} (g^2) \in C_t^0(\mathbb{R}; H_0^s(\mathbb{T})), \end{aligned}$$

by the algebra property of  $H^s$ ,  $s > 1/2$ . Therefore,

$$(26) \quad \mathbf{M}^{-1}e^{-\partial_x^3 t} \mathbf{M}g - e^{-\partial_x^3 t} g \in C_t^0(\mathbb{R}; H_0^{s+1}(\mathbb{T})).$$

Writing

$$w(t) - e^{-\partial_x^3 t} g = \mathbf{M}^{-1}u(t) - \mathbf{M}^{-1}e^{-\partial_x^3 t} \mathbf{M}g + \mathbf{M}^{-1}e^{-\partial_x^3 t} \mathbf{M}g - e^{-\partial_x^3 t} g,$$

and using (25) and (26), we obtain

$$w(t) - e^{-\partial_x^3 t} g \in C_t^0(\mathbb{R}; H_0^{s_1}(\mathbb{T})), \quad s_1 < \min(3s - 1, s + 1).$$

To obtain the result for  $v$ , let  $F(x, t) := v(x, t) - (e^{Lt}g)(x)$ . Note that

$$F(x - 6\mu t, t) = w(x, t) - (e^{Lt}g)(x - 6\mu t) = w(x, t) - (e^{-\partial_x^3 t}g)(x) \in C_t^0(\mathbb{R}; H_0^{s_1}(\mathbb{T})),$$

for  $s_1 < \min(3s - 1, s + 1)$ . This yields the statement for  $v$  by translation invariance of Sobolev spaces.  $\square$

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