A BILINEAR FOURIER EXTENSION THEOREM AND APPLICATIONS TO THE DISTANCE SET PROBLEM

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Abstract. In this paper, we obtain a weighted version of Tao's bilinear Fourier extension estimate for elliptic surfaces. This implies improved partial results in the direction of Falconer's distance set conjecture in dimensions $d \geq 3$.

1. Introduction

In [11], Tao proved the following bilinear Fourier extension estimate. Let $S = \{x \in \mathbf{R}^d : x_d = x_1^2 + ... + x_{d-1}^2\}$ and $d\sigma$ be the surface measure on S. Let $\widehat{\mu}$ denote the Fourier transform of the measure μ in $\mathbf{R}^d$,

$$\widehat{\mu}(\xi) = \int_{\mathbf{R}^d} e^{-2\pi i x \cdot \xi} d\mu(x), \quad \xi \in \mathbf{R}^d.$$

Theorem A. *Let $d \geq 2$. Let S_1, S_2 be compact subsets of S with $d(S_1, S_2) > 1$. Then for all $q > \frac{d+2}{d}$, we have*

$$\|\widehat{f_1 d\sigma}\,\widehat{f_2 d\sigma}\|_{L^q(\mathbf{R}^d)} \leq C_{q,d} \|f_1\|_{L^2(d\sigma)} \|f_2\|_{L^2(d\sigma)}, \tag{1}$$

for all $f_j \in L^2(d\sigma)$ supported in S_j, $j = 1, 2$.

This theorem is proved in [11] for $d \geq 3$. For $d = 2$, it has been known for a long time and is basically the Carleson-Sjölin Theorem [2]. Previously, in [15], Wolff obtained Theorem A for the light cone in general dimensions. Tao's proof relies on and extends the ideas in [15].

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We consider the following weighted version of the inequality (1). Fix $\alpha \in (0,d)$. Suppose $H : \mathbf{R}^d \to \mathbf{R}$ satisfies

$$\|H\|_\infty \leq 1, \tag{2}$$

$$\int_{B(x,r)} |H(u)|du \leq r^\alpha, \quad \forall x \in \mathbf{R}^d, \quad \forall r > 0. \tag{3}$$

For which q and α, the inequality

$$\|\widehat{f_1 d\sigma}\widehat{f_2 d\sigma}\|_{L^q(Hd\xi)} \leq C_{\alpha,q,d}\|f_1\|_{L^2(d\sigma)}\|f_2\|_{L^2(d\sigma)} \tag{4}$$

holds for all $f_j \in L^2(d\sigma)$ supported in S_j, $j = 1,2$?

Obviously, (2) and Theorem A imply that (4) holds for $q > \frac{d+2}{d}$. We improve this range of q for $\alpha < \frac{d+2}{2}$.

Theorem 1. *Let $d \geq 3$ and $\alpha \in (0,d)$. Assume that H satisfies* (2) *and* (3)*. Then, under the hypothesis of Theorem A,* (4) *holds for any* $q > q_0(\alpha,d) := \max(1, \min(\frac{4\alpha}{d+2\alpha-2}, \frac{d+2}{d}))$.

There is no reason for this theorem to be optimal. In fact, it should be possible to improve the range of q for each $\alpha \in (0,d)$. However, this theorem significantly improves the known estimates for the decay of L^2 spherical averages of the Fourier transform of fractal measures (see Section 3). Using this we obtain improved partial results in the direction of Falconer's distance set conjecture in dimensions 3 and higher. Let E be a compact subset of $\mathbf{R}^d$. The distance set, $\Delta(E)$, of E is defined as

$$\Delta(E) = \{|x-y| : x, y \in E\}.$$

In [5], Falconer conjectured that:

Conjecture. *Let $d \geq 2$. Let E be a compact subset of $\mathbf{R}^d$. Then,*

$$\dim(E) > \frac{d}{2} \implies |\Delta(E)| > 0.$$

Here $|\cdot|$ is the Lebesgue measure and $\dim(\cdot)$ *is the Hausdorff dimension.*

Falconer's conjecture is open in every dimension. In [5], Falconer gave an example showing that $\frac{d}{2}$ in the conjecture is optimal and proved that $\dim(E) > \frac{d+1}{2}$ implies $|\Delta(E)| > 0$. Bourgain [1] improved this result in every dimension, and in particular proved that in $\mathbf{R}^2$, $\dim(E) > \frac{13}{9}$ suffices. Later, Wolff [14] proved that in $\mathbf{R}^2$, $\dim(E) > \frac{4}{3}$ suffices. This is still the best known result in $\mathbf{R}^2$. See [3] for a simplified proof of Wolff's theorem. In [4], using Theorem A the author proved that $\dim(E) > \frac{d(d+2)}{2(d+1)}$ suffices. See [14], [8], [3] and [4] for some variations and related results. In this paper, using (a variant of) Theorem 1, we prove

Theorem 2. *Let $d \geq 3$. Let E be a compact subset of $\mathbf{R}^d$ with*

$$\dim(E) > \frac{d}{2} + \frac{1}{3}.$$

Then $|\Delta(E)| > 0$.

Remark 1. Wolff's result in [14] and Theorem 2 relies on a method developed by Mattila [7, 8]. As it is noted in [14], $\frac{4}{3}$ is the best possible exponent (in $\mathbf{R}^2$) one can obtain using this method. In $\mathbf{R}^3$, the best possible exponent is $\frac{5}{3}$. However, it may be possible to prove Falconer's conjecture in dimensions $d \geq 4$ using Mattila's approach. In particular, it will be clear from the proof of Theorem 2 that the inequality (4) for $\alpha = d/2$ and for all $q > 1$, if true, implies Falconer's conjecture in $\mathbf{R}^d$, $d \geq 4$.

Remark 2. As in [4], the assertion of Theorem 2 can be extended to distance sets with respect to general metrics. Let K be a convex symmetric body. Assume that the boundary of K is smooth and has non-vanishing Gaussian curvature. Define $\Delta_K(E) = \{d_K(x, y) : x, y \in E\}$, where d_K is the distance induced by K. Then, the statement of Theorem 2 holds for Δ_K.

We prove Theorem 1 in Section 5. In Section 2, we describe some extensions of Theorem 1. In Section 3, we describe Mattila's approach and in Section 4, we prove Theorem 2.

List of notations

χ_A: characteristic function of the set A.
$B(x,r) := \{y : |x-y| < r\}$.
$d(A,B)$: the distance between the sets A and B.
$A_R(C) := \{x \in \mathbf{R}^d : ||x| - R| \le C\}$.
C: a constant which may vary from line to line.
$A \lesssim B$: $A \le CB$.
$A \approx B$: $A \lesssim B$ and $B \lesssim A$.
$A \ll B$: $A \le \frac{1}{C}B$, for some large constant C.
$|A|$: length of the vector A or the measure of the set A.

2. Some extensions and corollaries of Theorem 1

Following the remark on page 1381 of [11], one can easily extend Theorem 1 to more general elliptic surfaces. First let us recall the definition of elliptic surfaces from [13] and [4].

Definition 1. *We say $\phi : B(0,1) \subset \mathbf{R}^{d-1} \to \mathbf{R}$ is an (M, ε_0)-elliptic phase if ϕ satisfies*
i) $\|\phi\|_{C^\infty} < M$,
ii) $\phi(0) = \nabla\phi(0) = 0$, and
iii) For all $x \in B(0,1)$, all eigenvalues of the Hessian $\phi_{x_i x_j}(x)$ lie in $[1-\varepsilon_0, 1+\varepsilon_0]$.
We say S is an (M, ε_0)-elliptic surface if $S = \{(x,y) \in B(0,1) \times \mathbf{R} \subset \mathbf{R}^d : y = \phi(x)\}$ for some (M, ε_0)-elliptic phase ϕ.

We recall the following properties of elliptic phases (see, e.g., [13, 4]):
I) Let ϕ be an (M, ε_0)-elliptic phase and $B(x_0, \eta) \subset B(0,1)$. Let

$$\tilde{\phi}(x) := \frac{1}{\eta^2}\left(\phi(x\eta + x_0) - \phi(x_0) - \eta x \cdot \nabla\phi(x_0)\right), \quad x \in B(0,1).$$

Then $\tilde{\phi}$ is a $(C_d M, \varepsilon_0)$-elliptic phase.

II) Let S be a smooth compact submanifold of $\mathbf{R}^d$ with strictly positive principal curvatures. Note that for any $\varepsilon_0 > 0$ and for any $s \in S$ there is a neighborhood U_s of s and an affine bijection a_s of $\mathbf{R}^d$ such that $a_s(U_s)$ is an (M,ε_0)-elliptic surface, where M depends only on d, $\|\phi\|_{C^\infty}$ and the principal curvatures at s. Moreover, by using a partition of unity, we can write S as a union of affine images of finitely many (M,ε_0)-elliptic surfaces.

We have the following generalization of Theorem 1.

Theorem 3. *Let $d \geq 3$ and $\alpha \in (0,d)$. Let H be a function satisfying (2) and (3). For any $M > 0$, there exists $\varepsilon_0 > 0$ such that the following statement holds.*
Let S_1, S_2 be compact subsets of diameter ≈ 1 of an (M,ε_0)-elliptic surface in $\mathbf{R}^d$ with $d(S_1,S_2) > \frac{1}{100}$. Let σ be the Lebesgue measure on S. Then for all $q > q_0(\alpha,d)$, we have

$$\|\widehat{f_1\,d\sigma_1}\widehat{f_2\,d\sigma_2}\|_{L^q(\mathbf{R}^d,Hdx)} \leq C_{M,q,d}\|f_1\|_{L^2(S_1,d\sigma_1)}\|f_2\|_{L^2(S_2,d\sigma_2)}, \tag{5}$$

for all $f_j \in L^2(d\sigma)$ supported in S_j, $j=1,2$.

In the application to the distance set problem, we need the following corollary of this theorem. Recall that

Definition 2. *A compactly supported probability measure μ is called α-dimensional if it satisfies*

$$\mu(B(x,r)) \leq C_\mu r^\alpha, \ \ \forall r > 0, \forall x \in \mathbf{R}^d. \tag{6}$$

Corollary 1. *Let μ be an α-dimensional measure. Let $\beta > 0$ and $\beta R^{-1/2} \lesssim \eta \lesssim 1$. Let I_1, I_2 be subsets of $A_R(\beta) = \{x \in \mathbf{R} : ||x| - R| < \beta\}$, satisfying*

$$\mathrm{diam}(I_j) \approx R\eta, \ \ j=1,2, \quad d(I_1,I_2) \approx R\eta.$$

Then for any $q > q_0(\alpha,d)$

$$\|\widehat{f_1}\widehat{f_2}\|_{L^q(d\mu)} \lesssim \beta (R\eta)^{d-1-\frac{\alpha}{q}}\eta^{-\frac{1}{q}}\|f_1\|_2\|f_2\|_2, \tag{7}$$

for any functions f_j supported in I_j, $j=1,2$.

We need the following version of the uncertainty principle in the proof of the corollary. For a proof see, e.g., [4] and [16, Chapter 5]. Let φ be a Schwartz function satisfying

$$\varphi(\xi)=1, \text{ for } |\xi|<2 \text{ and } \varphi(\xi)=0, \text{ for } |\xi|>4.$$

For each ball $D\subset\mathbf{R}^d$ fix an affine bijection a_D of $\mathbf{R}^d$ which maps D to $B(0,1)$. Let $\varphi_D:=\varphi\circ a_D$.

Lemma 2.1. *Let μ be an α-dimensional measure in $\mathbf{R}^d$. Let D be a ball of radius s in $\mathbf{R}^d$. Then the function $\mu_D:=|\varphi_D^\vee|*\mu$ satisfies*
i) $\|\mu_D\|_\infty\lesssim s^{d-\alpha}$,
ii) $\|\mu_D\|_1\lesssim 1$,
ii) $\mu_D(\mathcal{B}):=\int_{\mathcal{B}}\mu_D(y)\,dy\lesssim r^\alpha$, for any ball $\mathcal{B}$ of radius $r\geq 100s^{-1}$.

Proof of Corollary 1. Note that f_1*f_2 is contained in a ball D of radius $\approx R\eta$. Therefore

$$\begin{aligned}\|\widehat{f}_1\widehat{f}_2\|_{L^q(d\mu)}&=\|(\widehat{f}_1\widehat{f}_2)*\varphi_D^\vee\|_{L^q(d\mu)}\\&\lesssim\|\widehat{f}_1\widehat{f}_2\|_{L^q(|\varphi_D^\vee|*d\mu)}\|\varphi_D^\vee\|_1^{1/q'}\\&\lesssim\|\widehat{f}_1\widehat{f}_2\|_{L^q(\mu_D)}.\end{aligned}\tag{8}$$

Let e be the unit vector in the direction of the center of mass of $I_1\cup I_2$. Let $\{e_1=e,e_2,...,e_d\}$ be an orthogonal basis for $\mathbf{R}^d$. Let $T:\mathbf{R}^d\to\mathbf{R}^d$ be the linear map which satisfies

$$T(e_1)=\frac{1}{R\eta^2}e_1,\quad T(e_j)=\frac{1}{R\eta}e_j,\ j=2,3,...,d,$$

In view of I) and II) above, $C_j=TI_j$ is contained in $\approx\frac{\beta}{R\eta^2}$-neighborhood of an affine image of a surface S_j, $j=1,2$, where the surfaces S_1, S_2 satisfy the hypothesis of Theorem 3 (with M independent of R,η,I_1,I_2).

Let $g_j(x)=f_j(T^{-1}x)$, $j=1,2$. Note that g_j is supported in C_j, $j=1,2$. We have

$$\widehat{f}_j(\xi)=\frac{1}{\det(T)}\widehat{g}_j(T^{-1}(\xi))=(R\eta)^d\eta\widehat{g}_j(T^{-1}\xi),\ \ j=1,2.$$

Therefore,

$$\|\widehat{f_1}\widehat{f_2}\|_{L^q(\mu_D)} = (R\eta)^{2d}\eta^2 \left[\int |\widehat{g_1}(T^{-1}x)\widehat{g_2}(T^{-1}x)|^q \mu_D(x)dx\right]^{1/q} \tag{9}$$
$$= (R\eta)^{2d-\frac{d}{q}}\eta^{2-\frac{1}{q}} \left[\int |\widehat{g_1}(x)\widehat{g_2}(x)|^q \mu_D(Tx)dx\right]^{1/q}$$
$$= (R\eta)^{2d-\frac{d}{q}}\eta^{2-\frac{1}{q}}(R\eta)^{\frac{d-\alpha}{q}}\|\widehat{g_1}\widehat{g_2}\|_{L^q(H\,dx)},$$

where $H(x) = (R\eta)^{\alpha-d}\mu_D(Tx)$. Using Lemma 2.1, it is easy to see that H satisfies the conditions (2) and (3) (possibly with a constant other than 1 which can be scaled out). Since g_j is supported in C_j, using Theorem 3 we obtain

$$\|\widehat{g_1}\widehat{g_2}\|_{L^q(Hdx)} \lesssim \frac{\beta}{R\eta^2}\|g_1\|_2\|g_2\|_2. \tag{10}$$

We also have

$$\|g_j\|_2 = (R\eta)^{-\frac{d}{2}}\eta^{-\frac{1}{2}}\|f_j\|_2, \ \ j = 1, 2. \tag{11}$$

Using (8), (9), (10) and (11), we have

$$\|\widehat{f_1}\widehat{f_2}\|_{L^q(d\mu)} \lesssim (R\eta)^{2d-\frac{d}{q}}\eta^{2-\frac{1}{q}}(R\eta)^{\frac{d-\alpha}{q}}\frac{\beta}{R\eta^2}(R\eta)^{-d}\eta^{-1}\|f_1\|_2\|f_2\|_2$$
$$= \beta(R\eta)^{d-1-\frac{\alpha}{q}}\eta^{-\frac{1}{q}}\|f_1\|_2\|f_2\|_2.$$

□

3. Application to the distance set problem

In [7] (also see ([16, 8]), Mattila developed a method to attack the distance set problem. Mattila's approach was used in [7, 1, 14, 6, 4]. We refer the reader to [14] and [4] for the following version of Mattila's theorem.

Theorem 4. *Fix $\alpha \in [\frac{d}{2}, \frac{d+1}{2}]$ and $q_0 \in [1, 2]$ such that $\alpha(1 + \frac{1}{q_0}) \geq d$. Assume that for all $q > q_0$, for all α-dimensional measures μ, for all $R > 1$ and for all f supported in $A_R(1)$, we have*

$$\left|\int f^\vee(u)\,d\mu(u)\right| \leq C_{q,\mu}R^{\frac{d-1}{2}-\frac{\alpha}{2q}}\|f\|_2, \tag{12}$$

where $f^\vee$ is the inverse Fourier transform of f. Then Falconer's conjecture holds for α, i.e.

$$\dim(E) > \alpha \Rightarrow |\Delta(E)| > 0.$$

In light of Theorem 4, Theorem 2 is a corollary of the following

Theorem 5. *Let $\alpha \in (0, d)$ and $q > q_0(\alpha, d)$. For all α-dimensional measures μ, for all $R > 1$ and for all f supported in $A_R(1)$, (12) holds.*

Like Theorem 2, Theorem 5 was first proved in [14] for $d = 2$.

Remark 3. By duality and the uncertainty principle (see [4]), the inequality (12) implies that for every $\beta < \frac{\alpha}{2q}$

$$\|\widehat{\mu}(R\cdot)\|_{L^2(S^{d-1})} \lesssim R^{-\beta}.$$

In fact, one can easily keep track of the constant $C_{q,\mu}$ in (12) and obtain the statement

$$\|\widehat{\mu}(R\cdot)\|_{L^2(S^{d-1})} \leq C_{\alpha,\beta} R^{-\beta} \sqrt{I_\alpha(\mu)}, \tag{13}$$

for any $\beta < \frac{\alpha}{2q}$ (see [14, 3]). Here $I_\alpha(\mu)$ is the α-dimensional energy of the measure μ,

$$I_\alpha(\mu) := \int\int \frac{d\mu(x)d\mu(y)}{|x-y|^\alpha} = C_{\alpha,d} \int \frac{|\widehat{\mu}(\xi)|^2}{|\xi|^{d-\alpha}} d\xi.$$

Combining the result of Theorem 5 with the previously known partial results [7, 9, 14, 10, 8, 3, 4], we see that the inequality (13) holds for every

$$\beta < \begin{cases} \frac{\alpha}{2}, & \alpha \in (0, \frac{d-1}{2}], \\ \frac{d-1}{4}, & \alpha \in [\frac{d-1}{2}, \frac{d}{2}], \\ \frac{d+2\alpha-2}{8}, & \alpha \in [\frac{d}{2}, \frac{d+2}{2}], \\ \frac{\alpha-1}{2}, & \alpha \in [\frac{d+2}{2}, d). \end{cases} \tag{14}$$

The range of β is optimal for each $\alpha \in (0, 2)$ for $d = 2$ (see, e.g., [9, 14, 3]). In higher dimensions, the range is optimal for $\alpha \leq \frac{d-1}{2}$ (see [9]). However, there is no reason to believe that the range is optimal for $\alpha > \frac{d-1}{2}$ and $d \geq 3$.

4. Proof of Theorem 5

The proof is same as the proof given in [4] except a minor change in the inequality (22) below. Fix $\alpha \in (0, d)$. Let f be supported in $A_R(1)$ with L^2 norm 1. Below, we prove that for each $q > q_0(\alpha, d)$

$$\|f^\vee\|_{L^2(d\mu)} \lesssim R^{\frac{d-1}{2}-\frac{\alpha}{2q}}. \tag{15}$$

(12) can be obtained from (15) using Cauchy-Schwarz inequality. As in [4], we use the bilinear approach. It suffices to prove (15) for functions f supported in a subset of $A_R(1)$ of diameter $\ll R$. Consider a dyadic decomposition of $A_R(1)$ into spherical caps, I, with dimensions $2\times 2^n \times ... \times 2^n$ for

$$R^{\frac{1}{2}} \ll 2^n \ll R.$$

We say I has sidelength 2^n and write $\ell(I) = 2^n$. The unique cap of sidelength 2^{n+1} which contains I is called the parent of I. Let I and J be caps with the same sidelength. We say I and J are related, $I \sim J$, if they are not adjacent but their parents are.

Let $f_I := f\chi_I$. As in [4], we have

$$\|f^\vee\|^2_{L^2(d\mu)} \leq \sum_{R^{\frac{1}{2}} \ll 2^n \ll R} \ \sum_{\ell(I)=2^n, I\sim J} \|f_I^\vee f_J^\vee\|_{L^1(d\mu)} + \sum_{I\in I_E} \|f_I^\vee\|^2_{L^2(d\mu)} \tag{16}$$
$$=: S_1 + S_2.$$

Here I_E is a set of dyadic caps with sidelengths $\approx R^{\frac{1}{2}}$ satisfying the finite overlapping property:

$$\Big\| \sum_{I\in I_E} \chi_I \Big\|_\infty \lesssim 1. \tag{17}$$

First, we obtain a bound for S_2. Since each $I \in I_E$ is contained in a ball D of radius $CR^{\frac{1}{2}}$, we have $f_I^\vee = f_I^\vee * \varphi_D^\vee$, ($\varphi_D$ is defined in the Section 2). Using this and Cauchy-Schwarz inequality, we have

$$|f_I^\vee| \leq (|f_I^\vee|^2 * |\varphi_D^\vee|)^{\frac{1}{2}} \|\varphi_D^\vee\|_1^{\frac{1}{2}} \lesssim (|f_I^\vee|^2 * |\varphi_D^\vee|)^{\frac{1}{2}}. \tag{18}$$

Using this, Fubini's theorem and Lemma 2.1, we obtain

(19)
$$\|f_I^\vee\|_{L^2(d\mu)}^2 \le \int |f_I^\vee(x)|^2(\mu * |\varphi_D^\vee|)(x)dx \lesssim \|f_I^\vee\|_2^2 R^{\frac{d-\alpha}{2}} = \|f_I\|_2^2 R^{\frac{d-\alpha}{2}}.$$

Using (19) and (17), we obtain

$$S_2 = \sum_{I\in I_E} \|f_I^\vee\|_{L^2(d\mu)}^2 \lesssim R^{\frac{d-\alpha}{2}} \sum_{I\in I_E} \|f_I\|_2^2 \lesssim R^{\frac{d-\alpha}{2}} \|f\|_2^2 = R^{\frac{d-\alpha}{2}}.$$

This term is harmless since $\frac{d-\alpha}{2} \le d-1-\frac{\alpha}{q_0(\alpha,d)}$, for $\alpha \in (0,d)$.

In the remaining part of the paper we prove that for $q > q_0(\alpha,d)$, $S_1 \lesssim R^{d-1-\frac{\alpha}{q}}$. By a standard L^2-orthogonality argument (see e.g. [13, 15, 3, 4]), it suffices to prove that for each $q > q_0(\alpha,d)$, for each n and $I \sim J$ with $|I| = |J| = 2^n$

$$\|f_I^\vee f_J^\vee\|_{L^1(d\mu)} \le C_{\alpha,q,d} R^{d-1-\frac{\alpha}{q}} \|f_I\|_2 \|f_J\|_2. \tag{20}$$

Let e be the unit vector which is in the direction of the center of mass of $I \cup J$. Consider a tiling of $\mathbf{R}^d$ with rectangles P of dimensions $100 \times 100\frac{2^n}{R} \times ... \times 100\frac{2^n}{R}$, the long axis being in the direction e. For each P, let a_P be an affine bijection from $\mathbf{R}^d$ to $\mathbf{R}^d$ which maps P to the unit cube. Let ϕ be a Schwartz function satisfying

$$\phi(x) \ge \chi_{B(0,1)}(x),\ x \in \mathbf{R}, \text{ and } \quad \text{supp}(\widehat{\phi}) \subset B(0,1). \tag{21}$$

Let $\phi_P := \phi \circ a_P$ and $f_{I,P} := \widehat{f_I^\vee \phi_P}$. Using (21) and the fact that the rectangles P tile $\mathbf{R}^d$, we obtain

$$\begin{aligned} \|f_I^\vee f_J^\vee\|_{L^1(d\mu)} &\lesssim \sum_P \int |f_{I,P}^\vee(x) f_{J,P}^\vee(x)| \phi_P(x) d\mu(x) \\ &\lesssim \sum_P \|f_{I,P}^\vee f_{J,P}^\vee\|_{L^q(\mu)} \|\phi_P\|_{L^1(\mu)}^{1/q'}, \end{aligned} \tag{22}$$

where $q' = \frac{q}{q-1}$.

To estimate $\|f_{I,P}^\vee f_{J,P}^\vee\|_{L^q(\mu)}$, we use Corollary 1 of Theorem 1. Let I_P be the support of $f_{I,P}$. Note that I_P is contained in $I + \text{supp}(\widehat{\phi_P}) \subset I + P_{dual}$, where P_{dual} is the dual of P centered at the origin. We have

Lemma 4.1. *$I + P_{dual}$ is contained in a spherical cap of dimensions $10 \times \frac{11}{10}2^n \times ... \times \frac{11}{10}2^n$ in $A_R(10)$ which contains I.*

See [4] for the elementary proof. Using Lemma 4.1 for I and J, we see that I_P and J_P have diameter $\approx 2^n$; they are contained in $A_R(10)$ and $d(I_P, J_P) \approx 2^n$. Therefore, Corollary 1 implies that

$$\begin{aligned} \|f_{I,P}^\vee f_{J,P}^\vee\|_{L^q(\mu)} &\lesssim 2^{n(d-1-\frac{\alpha}{q})}\left[\frac{2^n}{R}\right]^{-\frac{1}{q}} \|f_{I,P}\|_2 \, \|f_{J,P}\|_2 \\ &= R^{\frac{1}{q}} 2^{n(d-1-\frac{1}{q}-\frac{\alpha}{q})} \|f_{I,P}\|_2 \, \|f_{J,P}\|_2. \end{aligned} \tag{23}$$

Now, we estimate $\|\phi_P\|_{L^1(\mu)}$. Using the Schwartz decay of ϕ_P, we have

$$\|\phi_P\|_{L^1(\mu)} \leq \sum_{j=1}^{\infty} 2^{-Mj} \int \chi_{2^j P}(x) \, d\mu(x).$$

Note that $2^j P$ can be covered by $\approx \frac{R}{2^n}$ balls of radius $\approx \frac{2^j 2^n}{R}$. Since μ is α dimensional, we get

$$\|\phi_P\|_{L^1(\mu)} \lesssim \sum_{j=1}^{\infty} 2^{-\frac{Mj}{2}} 2^{n\alpha - n} R^{1-\alpha} \lesssim 2^{n\alpha-n} R^{1-\alpha}. \tag{24}$$

Using (22), (23), (24) and then Cauchy-Schwarz inequality, we get

$$\begin{aligned} \|f_I^\vee f_J^\vee\|_{L^1(d\mu)} &\lesssim R^{1-\frac{\alpha}{q'}} 2^{n(\alpha(1-\frac{2}{q})+d-2)} \sum_P \|f_{I,P}\|_2 \|f_{J,P}\|_2 \\ &\lesssim R^{1-\frac{\alpha}{q'}} 2^{n(\alpha(1-\frac{2}{q})+d-2)} \left[\sum_P \|f_{I,P}\|_2^2\right]^{\frac{1}{2}} \left[\sum_P \|f_{J,P}\|_2^2\right]^{\frac{1}{2}} \end{aligned}$$

Using the Schwartz decay of ϕ, the fact that the rectangles P tile $\mathbf{R}^d$ and Plancherel formula, we get

$$\|f_I^\vee f_J^\vee\|_{L^1(d\mu)} \lesssim R^{1-\frac{\alpha}{q'}} 2^{n(\alpha(1-\frac{2}{q})+d-2)} \|f_I\|_2 \|f_J\|_2. \tag{25}$$

The exponent of 2^n in (25) is non-negative and $2^n \lesssim R$. Therefore

$$\begin{aligned} \|f_I^\vee f_J^\vee\|_{L^1(d\mu)} &\lesssim R^{1-\frac{\alpha}{q'}} R^{\alpha(1-\frac{2}{q})+d-2} \|f_I\|_2 \|f_J\|_2 \\ &= R^{d-1-\frac{\alpha}{q}} \|f_I\|_2 \|f_J\|_2. \end{aligned} \tag{26}$$

This finishes the proof of Theorem 5.

5. Proof of Theorem 1

The proof of Theorem A is quite technical and lengthy. The proof of Theorem 1 is a very simple modification of this proof. Therefore, we just give a short sketch of the argument. We warn the reader that in [11] the letter n is used for dimension. We use the letter d for dimension and $d = n + 1$. We also note that the "epsilon-removal" lemma in [12, Lemma 2.4] which was used in [11] to reduce the proof to a localized restriction estimate remains valid in our case. Thus, it suffices to prove the following localized version of (4) for each $\eta > 0$

(27)
$$\|\widehat{f_1 d\sigma}\widehat{f_2 d\sigma}\|_{L^{q_0(\alpha,d)}(B(0,R),Hd\xi)} \leq C_{\eta,\alpha,d} R^\eta \|f_1\|_{L^2(d\sigma)} \|f_2\|_{L^2(d\sigma)}, \;\; \forall R > 1.$$

In fact, Theorem 2 can be proved using only the localized version (27) since we don't prove an endpoint result.

Fix $\alpha \leq \frac{d+2}{2}$, and let $q_0 = q_0(\alpha, d)$. As in [15], (27) is proved by induction on η. It is easy to see that (27) holds for each $R > 0$ and for each ball B of radius R if $\eta \geq \alpha/q_0$. Now, we prove that if (27) holds for some $\eta > 0$ (and for each R and B), then it also holds for $\max((1-\delta)\eta, C\delta) + C\varepsilon$ for all $0 < \delta, \varepsilon < 1$. This implies (27) for each $\eta > 0$.

The first step in the proof is a standard wave packet decomposition in scale R (see [11, Lemma4.1])

$$\widehat{f_j d\sigma}(\xi) = \sum_{T_j} c_{T_j} \phi_{T_j}(\xi), \quad \xi \in B, \;\; j = 1, 2.$$

Here, T_j are $R^{1/2}$-separated $R^{1/2} \times ... \times R^{1/2} \times R$ tubes. c_{T_j} are constants and ϕ_{T_j} are Knapp examples. Namely, each ϕ_{T_j} is essentially supported in the tube T_j with a Schwartz decay away from T_j and $\phi_{T_j}^\vee$ is supported in a dual rectangle of T_j of dimensions $R^{-1/2} \times ... \times R^{-1/2} \times R^{-1}$ which is contained in $O(R^{-1})$ neighborhood of the surface S_j. The tubes T_j are called S_j-tubes. In [11], the wave packets are normalized so that

$$\|\phi_{T_j}\|_2 \approx R^{1/2}, \qquad \sum_{T_j} |c_{T_j}|^2 \lesssim \|f_j\|^2_{L^2(d\sigma)}. \tag{28}$$

Moreover, the functions ϕ_{T_j} are almost orthogonal in the sense that

$$\Big\|\sum \phi_{T_j}\Big\|_2^2 \lesssim \sum \|\phi_{T_j}\|_2^2. \tag{29}$$

By dyadic pigeonholing and normalization, we can assume that each c_j is either 0 or 1. Therefore, it suffices to prove that

$$\Big\|\sum_{T_1\in\mathbf{T}_1}\sum_{T_2\in\mathbf{T}_2}\phi_{T_1}\phi_{T_2}\Big\|_{L^{q_0}(B,Hd\xi)} \lesssim R^{\max((1-\delta)\eta,C\delta)+C\varepsilon}(\#\mathbf{T}_1)^{1/2}(\#\mathbf{T}_2)^{1/2}, \tag{30}$$

for all collections $\mathbf{T}_j$ of S_j-tubes and for each $0<\delta,\varepsilon<1$. Cover B by a collection $\mathcal{B}$ of $O(R^{C\delta})$ finitely overlapping balls of radius $R^{1-\delta}$. We need the following lemma which summarizes the main part of the argument in [11].

Lemma 5.1. [11] *There is a relation $\sim$ between the balls $Q\in\mathcal{B}$ and the tubes in $\mathbf{T}_1\cup\mathbf{T}_2$ such that for each $T\in\mathbf{T}_1\cup\mathbf{T}_2$*

$$\#\{Q\in\mathcal{B} : T\sim Q\}\le C_\varepsilon R^\varepsilon, \tag{31}$$

and the following L^2 estimate holds

$$\Big\|\sum_{(T_1,T_2)\in Q^{\not\sim}}\phi_{T_1}\phi_{T_2}\Big\|_{L^2(Q)} \lesssim R^{C\delta+C\varepsilon}R^{-(d-2)/4}(\#\mathbf{T}_1)^{1/2}(\#\mathbf{T}_2)^{1/2}, \tag{32}$$

where $Q^{\not\sim}=\{(T_1,T_2)\in\mathbf{T}_1\times\mathbf{T}_2 : T_1\not\sim Q$ or $T_2\not\sim Q\}$.

Remark 4. We use this lemma without any modification in the proof. It may be possible to get an estimate for $\|\sum_{(T_1,T_2)\in Q^{\not\sim}}\phi_{T_1}\phi_{T_2}\|_{L^2(Q,Hd\xi)}$ which is better than (32). This would improve the range of q in Theorem 1 and it may give a better partial result in the direction of Falconer's distance set problem.

We have

$$\begin{aligned}\Big\| \sum_{T_1\in\mathbf{T}_1}\sum_{T_2\in\mathbf{T}_2}\phi_{T_1}\phi_{T_2}\Big\|_{L^{q_0}(B,Hd\xi)} &\leq \sum_{Q\in\mathcal{B}}\Big\|\sum_{T_1\in\mathbf{T}_1}\sum_{T_2\in\mathbf{T}_2}\phi_{T_1}\phi_{T_2}\Big\|_{L^{q_0}(Q,Hd\xi)}\\ &\leq \sum_{Q\in\mathcal{B}}\Big\|\sum_{T_1\sim Q}\sum_{T_2\sim Q}\phi_{T_1}\phi_{T_2}\Big\|_{L^{q_0}(Q,Hd\xi)}\\ &\quad+\sum_{Q\in\mathcal{B}}\Big\|\sum_{(T_1,T_2)\in Q^{\not\sim}}\phi_{T_1}\phi_{T_2}\Big\|_{L^{q_0}(Q,Hd\xi)}\\ &=: I_1+I_2.\end{aligned}$$

The estimate for I_1 follows from the induction hypothesis. Remember that Q is a $R^{1-\delta}$-ball, and ϕ_{T_j} is supported in $O(R^{-1})$ neighborhood of S_j, $j=1,2$. Therefore

$$\begin{aligned}I_1 &\lesssim R^{-1}R^{(1-\delta)\eta}\sum_{Q\in\mathcal{B}}\Big\|\sum_{T_1\sim Q}\phi_{T_1}\Big\|_2\Big\|\sum_{T_2\sim Q}\phi_{T_2}\Big\|_2\\ &\lesssim R^{-1}R^{(1-\delta)\eta}\sum_{Q\in\mathcal{B}}\Big(\sum_{T_1\sim Q}\|\phi_{T_1}\|_2^2\sum_{T_2\sim Q}\|\phi_{T_2}\|_2^2\Big)^{1/2}\\ &\lesssim R^{(1-\delta)\eta}\sum_{Q\in\mathcal{B}}(\#\{T_1\in\mathbf{T}_1:T_1\sim Q\}\#\{T_2\in\mathbf{T}_2:T_2\sim Q\})^{1/2}\\ &\lesssim R^{(1-\delta)\eta+C\varepsilon}(\#\mathbf{T}_1)^{1/2}(\#\mathbf{T}_2)^{1/2}.\end{aligned}$$

The second inequality follows from (29), the third from (28) and the last from (31) and Cauchy-Schwarz. Now, we estimate I_2. This is the only part of the proof which differs from the proof in [11]. Let

$$F_Q := \sum_{(T_1,T_2)\in Q^{\not\sim}}\phi_{T_1}\phi_{T_2}.$$

Using Hölder's inequality, (2) and (3), we have

$$\begin{aligned}I_2 &\lesssim \sum_{Q\in\mathcal{B}}\|F_Q\|_{L^2(Q,d\xi)}\left[\int_Q|H(\xi)|^{2/(2-q_0)}d\xi\right]^{\frac{1}{q_0}-\frac12}\\ &\leq \sum_{Q\in\mathcal{B}}\|F\|_{L^2(Q,d\xi)}\left[\int_Q|H(\xi)|d\xi\right]^{\frac{1}{q_0}-\frac12}\\ &\lesssim \sum_{Q\in\mathcal{B}}\|F\|_{L^2(Q,d\xi)}R^{\frac{\alpha}{q_0}-\frac{\alpha}{2}}.\end{aligned}\tag{33}$$

Using Lemma 5.1 and the definition of $q_0 = q_0(\alpha, d)$ for $\alpha \leq (d+2)/2$, we have

$$\begin{aligned}(33) &\lesssim R^{C\delta + C\varepsilon} R^{-(d-2)/4} R^{\frac{\alpha}{q_0} - \frac{\alpha}{2}} (\#\mathbf{T}_1)^{1/2} (\#\mathbf{T}_2)^{1/2} \\ &\lesssim R^{C\delta + C\varepsilon} (\#\mathbf{T}_1)^{1/2} (\#\mathbf{T}_2)^{1/2}.\end{aligned}$$

This finishes the proof of (30).

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