TALBOT EFFECT ON THE SPHERE AND TORUS FOR $d \geq 2$

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Abstract. We utilize exponential sum techniques to obtain upper and lower bounds for the fractal dimension of the graph of solutions to the linear Schrödinger equation on $\mathbb{S}^d$ and $\mathbb{T}^d$. Specifically for $\mathbb{S}^d$, we provide dimension bounds using both L^p estimates of Littlewood-Paley blocks, as well as assumptions on the Fourier coefficients. In the appendix, we present a slight improvement to the bilinear Strichartz estimate on $\mathbb{S}^2$ for functions supported on the zonal harmonics. We apply this to demonstrate an improved local well-posedness result for the zonal cubic NLS when $d = 2$, and a nonlinear smoothing estimate when $d \geq 2$. As a corollary of the nonlinear smoothing for solutions to the zonal cubic NLS, we find dimension bounds generalizing the results of [ErTz2] for solutions to the cubic NLS on $\mathbb{T}$. Additionally, we obtain several results on $\mathbb{T}^d$ generalizing the results of the $d = 1$ case.

Contents

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1. Introduction

In this paper, we investigate so called the Talbot effect and fractal solutions of the linear Schrödinger equation on certain compact manifolds $\mathcal{M} = \mathbb{S}^d$, or $\mathbb{T}^d$ for $d \geq 2$:

$$
\begin{cases} iu_t + \triangle_{\mathcal{M}} u = 0 \\ u(0,x) = f(x) \in L^2(\mathcal{M}), \end{cases} \tag{1}
$$

where $\triangle_{\mathcal{M}}$ is the Laplace-Beltrami operator on $\mathcal{M}$.

Many authors have studied the properties and dimension of the graph of the solution to the linear Schrödinger equation and other dispersive equiations on $\mathbb{T}$ with varying initial data; see, e.g., [Be, BeKl, BMS, Os, KaRo, Ro, ErTz1, ErTz2, OsCh, ChOl, HoVe, CET, Ve, ErTz3, OlP, OlTs]. The history of this line of inquiry starts with an optical experiment in 1836 where Talbot studied monochromatic light passing through a diffraction grating [Ta]. He observed there is a certain distance (now called the Talbot distance) at which the diffraction pattern reproduces the grating pattern. It was further remarked that the pattern appears to be a finite linear combination of the grating pattern at each rational multiple of the Talbot distance. This phenomenon has since been referred to as the Talbot effect. Berry and collaborators were among the first to carry out exact calculations and numerical works on the Talbot effect in [Be, BeKl, BLN]. In particular, in [BeKl] it was proved that at rational times the solution is a linear combination of finitely many translates of the initial data with the coefficients being Gauss sums, also see [TaM1, TaM2, OlP, OlTs]. This phenomenon is often called *quantization* in the literature. In [BeKl], the authors also observed that the solution at irrational times has a fractal profile. In particular for step function initial data at rational times one observes a step function, and a continuous but nowhere differentiable function with fractal dimension $\frac{3}{2}$ at irrational times[1]. Finally, it was conjectured that this phenomenon should occur in higher dimensions and even when there is a nonlinear perturbation, also see [ZWZX] for an experimental justification.

In the field of mathematics, Oskolkov proved in [Os, Proposition 14] that for bounded variation initial data, the solution of any dispersive PDE on $\mathbb{T}$ with polynomial dispersion relation is a continuous function of x at irrational times. Kapitanski and Rodniaski showed in [KaRo] the solution to the linear Schrödinger equation at irrational times is more regular in the Besov scale than at rational times. Rodniaski then used results in [KaRo] to justify Berry's conjecture, proving, that given initial data in $BV(\mathbb{T}) \setminus H^{1/2+}(\mathbb{T})$, the graph of the

[1]Recall that the fractal dimension (or upper Minkowski/box dimension) of a bounded set E is given by $\overline{\dim} E := \limsup_{\epsilon\to 0} \frac{\log \mathcal{N}(E,\epsilon)}{\log(1/\epsilon)}$, where $\mathcal{N}(E,\epsilon)$ is the minimum number of boxes of sidelength ϵ needed to cover E.

real and imaginary parts of the solution to (1) has fractal dimension 3/2 for almost every time.

This paper is motivated by the works by the first author, Tzirakis, and Shakan in [ErTz1, ErTz2, ErTz3, ErSh]. In these papers, the authors considered the case of polynomial and nonpolynomial dispersion relations, and proved several results on dimension bounds using exponential sum estimates on $\mathbb{T}$. In particular, they obtained fractal dimension bounds for the graph of the real and imaginary part of solutions with bounded variation initial data. In addition, the results on the dimension were extended to nonlinear counterparts via nonlinear smoothing estimates. As these works were done on $\mathbb{T}$, it is clearly desirable to obtain analogous estimates on more general compact manifolds. In the case of $\mathbb{T}^d$, one can easily extend the rational time quantization results on $\mathbb{T}$ to $\mathbb{T}^d$, [TaM1]. It is also clear that the results in [Os, Ro, ErTz1, ErTz2, ErTz3, ErSh] can be extended to $\mathbb{T}^d$ in the case when the initial data is a tensor function, establishing the existence of fractal solutions, and a dichotomy similar to the one on $\mathbb{T}$. On the other hand, extending the fractal behavior results to even just $\mathbb{S}^2$ or to more general functions on $\mathbb{T}^d$ is not as straightforward, and proving satisfactory dimension bounds on $\mathbb{S}^d$ for $d \geq 3$ turns out to be quite challenging. We note that in [TaM2], Taylor studied the Talbot effect for the Schrödinger propagator on $\mathbb{S}^d$ and obtained multiplier estimates at rational times. This is analogous to the quantization behavior at rational times on the torus since finite linear combinations of translations are bounded operators on all L^p spaces. Also see [HaLo] and [ChSa] for various results on the quantization on $\mathbb{S}^2$. In this paper, by establishing the existence of fractal solutions at irrational times, we obtain a dichotomy on $\mathbb{S}^d$ similar to the one on $\mathbb{T}$, or $\mathbb{T}^d$.

To study this problem, as in the earlier papers, we will utilize Besov spaces, $B^s_{p,\infty}$, defined by the norm $\|f\|_{B^\gamma_{p,\infty}} := \sup\{N^\gamma \|f_N\|_{L^p} : N \geq 1, \text{ dyadic}\}$, $1 \leq p \leq \infty$, where f_N is the Littlewood-Paley projection to frequencies $\approx N$. Recall that for $0 < \gamma < 1$, $C^\gamma(\mathbb{T})$ coincides with $B^\gamma_{\infty,\infty}(\mathbb{T})$, and that if $f : \mathbb{T} \to \mathbb{R}$ is in C^γ, then the graph of f has fractal dimension $D \leq 2 - \gamma$. For lower bounds we have the following result of Deliu and Jawerth [DeJa] (also see [ErTz3, Theorem 2.24]): Fix $\gamma \in [0, 1]$. The graph of a continuous function $f : \mathbb{T} \to \mathbb{R}$ has fractal dimension $D \geq 2 - \gamma$ provided that $f \notin B^\gamma_{1,\infty}$. Analogous results hold for $\mathbb{T}^d$ and $\mathbb{S}^d$; see Theorem 2.4 and Theorem 2.8 below for the case of $\mathbb{S}^d$.

Before moving on to the statements of the results, we first take a moment to discuss generalizations of the $BV(\mathbb{T})$ requirement of [Os, Ro]. Specifically, $f \in BV(\mathbb{T})$ implies that $\widehat{f}(n) = \frac{1}{in}\widehat{df}(n)$, which leads to, for $1 \leq p \leq 2$, $\|f_N\|_{L^p_x(\mathbb{T})} \lesssim N^{-\frac{1}{p}}$. It follows then that a natural generalization would be the requirement that $\|f_N\|_{L^p} \lesssim N^{-\frac{d}{2}-s}$ for some $p \geq 1$, and $s \geq 0$. This is the approach taken for Theorems 1.1 and 6.1. On the other hand, $f \in BV(\mathbb{T})$ also implies that $|\widehat{f}(n)| \lesssim n^{-1}$ and additional bound on the differences up to a phase. Thus, the next possible generalization is to require decay on the Fourier coefficients, which is the approach of Theorems 1.3, 3.8, and 6.4.

In particular, when $\mathcal{M}$ is understood, we define $\dim_t(f)$ to be the maximum of the fractal dimensions of the real and imaginary parts of $u(\cdot,t)$, the solution to (1) emanating from f at time t; see (15) for $\mathbb{S}^d$. Using this notation, we show in Section 3 bounds on $\dim_t(f)$ depending on the L^p norms of the Littlewood-Paley pieces, f_N, of f on $\mathbb{S}^d$.

Theorem 1.1. *Let $f:\mathbb{S}^d\to\mathbb{R}$, $1<p\le 2$, $s>\max\{\frac{d}{p}-\frac{d+1}{2},0\}$. Assume that $\|f_N\|_{L^p}\lesssim N^{-\left(\frac{d}{2}+s\right)}$. Then for almost all t, the solution $u(\cdot,t)$ to (1) is in $C^{\gamma-}$ for $\gamma=\min\{s,s+\frac{d+1}{2}-\frac{d}{p},1\}$. Hence for almost all t, $\dim_t(f)\le d+1-\gamma$, where $\gamma=\min(s,s+\frac{d+1}{2}-\frac{d}{p},1)$.*

In the case $d=2$, if $f\notin H^{s+2-\frac{2}{p}+}(\mathbb{S}^2)$ in addition to the hypothesis above, then

$$\dim_t(f)\ge\max\left(\tfrac{3}{4}+\tfrac{2}{p}-s,2\right).$$

We only have the lower bound in the case $d=2$ because it relies on a Strichartz estimate which is not strong enough for this purpose when $d>2$.

Corollary 1.2. *Let $p=\frac{2d}{d+1}$ and assume $\|f_N\|_{L^p}\lesssim N^{-d/p}$. Then for almost all t, $u(t,x)\in C^{\frac{1}{2}-}$, and hence $\dim_t(f)\le d+\frac{1}{2}$.*

As the above formulation is not easy to use for specific f, we also seek generalizations in terms of the Fourier coefficients of f. In Section 3.1, we specialize to the case $d=2$ and provide bounds for $\dim_t(f)$ using only information on the Fourier coefficients in the specific cases that f is supported on the Zonal harmonics, $Y_n:=Y_n^0$, in $\mathbb{S}^2$:

Theorem 1.3. *Let $f(\theta,\phi)=\sum_{n=0}^{\infty}a_n\,Y_n(\theta,\phi)$. If for some $1<p<2$ we have*

$$|a_n|\lesssim\tfrac{1}{n^p},\ \ \textit{and}\quad |a_n-a_{n-1}|\lesssim\tfrac{1}{n^{p+1}}$$

for all $n\in\mathbb{N}\cup\{0\}$, then for almost all t, $u(x,t)\in C^{(p-1)-}$, and hence $\dim_t(f)\le 4-p$.

If, in addition, $f\notin H^{p-\frac{1}{2}+}(\mathbb{S}^2)$, then we also find $\dim_t(f)\ge\max\left(\frac{7}{2}-p,2\right)$.

We additionally extend this result to the case when f is supported on $\{Y_n^k\}_n$ for k fixed in Lemma 3.7, and the case when f is supported on Gaussian beams, $\{Y_n^n\}_n$, in Theorem 3.8.

In addition, we also note the following Corollary, whose proof is immediate from the methods in Theorem 1.3 and is stated separately from Theorem 1.3 due to the work in Appendix 5.

Corollary 1.4. *Suppose that $\frac{d}{2}<p<\frac{d}{2}+1$ and $\{a_n\}$ satisfies*

$$|a_n|\lesssim\tfrac{1}{n^p},\ \ \textit{and}\quad |a_n-a_{n-1}|\lesssim\tfrac{1}{n^{p+1}}.$$

Then if $f \in \mathbb{S}^d$ is supported on the zonal harmonics with Fourier coefficients $\{a_n\}_n$, then for almost all t, $u \in C^{p-\frac{d}{2}+}$ and $\dim_t(f) \le (d+1) - \left(p - \frac{d}{2}\right)$.

In Appendix 5, we provide an improved estimate on products of zonal harmonics on $\mathbb{S}^2$ which will be useful in obtaining lower bounds in Section 3.1. As a consequence of this estimate and the symmetries of the cubic nonlinear Schrödinger equation, we obtain both local well-posedness to (60) for $s > 0$ in Theorem 5.3 and nonlinear smoothing[2] in Theorem 1.5 for the class of functions supported on the zonal harmonics.

Theorem 1.5. *Let $d \ge 2$, $s > \frac{d-2}{2}$, and*

$$0 \le \varepsilon < \min\left(\tfrac{1}{2}\left(s - \tfrac{d-2}{2}\right), 1\right).$$

For $u_0 \in \mathcal{Z}^s(\mathbb{S}^d)$, let u denote the solution to (27) emanating from u_0 with local existence time $T > 0$. Then, letting (see (38) below)

$$\gamma(t;u) = \frac{2}{\pi\omega_d} \sum_{k,\ell} \overline{\widehat{u}_k}(t)\widehat{u}_\ell(t) \int_0^\pi Y_k(\theta)Y_\ell(\theta)\, d\theta, \tag{2}$$

we have

$$u - e^{it\triangle_{\mathbb{S}^d} \mp i\int_0^t \gamma(s;u)\,ds} u_0 \in C_t^0\left([0,T), H_x^{s+\varepsilon}\right).$$

In particular, for all $0 < t < T$,

$$\inf_{\theta\in\mathbb{R}} \left\| u - e^{i\theta} e^{it\triangle_{\mathbb{S}^d}} u_0 \right\|_{H_x^{s+\varepsilon}} \lesssim C_{\|u_0\|_{H^s}}.$$

We then use this nonlinear smoothing to show our final theorem on the cubic NLS posed on zonal functions on $\mathbb{S}^d$ for $d \ge 2$:

Theorem 1.6. *Let $d \ge 2$ and f satisfy the hypothesis of Theorem 1.4 for some $\frac{d+1}{2} < p < \frac{d+2}{2}$ and let u denote the solution to (27) emanating from f. Then $f \in \mathcal{Z}^{p-1/2-} := H^{p-1/2-}(\mathbb{S}^d) \cap$ span$\{Y_n : n \in \mathbb{N}\}$ and*

$$\dim_t(f) \le (d+1) - \left(p - \tfrac{d}{2}\right).$$

Finally, we offer several extensions of the results from [ErSh] to $\mathbb{T}^d$ in Appendix 6. Of particular interest is the statement of the result of [Ro] for Vitali BV functions (for definitions and a survey, see [AdCl2, AdCl, Za]), and Theorems 6.8 and 6.9 on the graphs of solutions emenating from characteristic functions of polygons and polytopes.

We finish the introduction with some notation. We say that $f \lesssim g$ if there is a $C > 0$ so that $f \le Cg$, and also denote $a+$ to be $a + \varepsilon$ for all $\varepsilon > 0$, with implicit constants that will depend on ε. We also define the bracket $\langle\cdot\rangle := (1 + |\cdot|^2)^{1/2}$.

[2]See Section 4 for an introduction to nonlinear smoothing and motivation for the statement of Theorem 1.5.

2. Harmonic Analysis on the Sphere

We start with a discussion of spherical convolution of $f : \mathbb{S}^d \to \mathbb{C}$ and $g : [-1,1] \to \mathbb{C}$, both continuous, say:

$$f * g(x) := \frac{1}{\omega_d} \int_{\mathbb{S}^d} f(y) g(\langle x, y\rangle)\, d\sigma(y), \quad x \in \mathbb{S}^d,$$

where $d\sigma$ is the surface measure and ω_d the surface volume of $\mathbb{S}^d$, and $\langle x, y\rangle$ is the $\mathbb{R}^{d+1}$ inner product of $x, y \in \mathbb{S}^d$. Note that

$$\|g(\langle x, \cdot\rangle)\|^r_{L^r(\mathbb{S}^d)} = \frac{1}{\omega_d} \int_{\mathbb{S}^d} |g(\langle x, y\rangle)|^r \; d\sigma(y) = \frac{\omega_{d-1}}{\omega_d} \int_{-1}^{1} |g(\tau)|^r (1-\tau^2)^{\frac{d-2}{2}} \; d\tau$$

is independent of $x \in \mathbb{S}^d$ (and similarly if we take the norm in the x variable for fixed y). Therefore, we define

$$(3) \quad \|g\|_{L^r_w([-1,1])} := \|g(\langle e_{d+1}, \cdot\rangle)\|_{L^r(\mathbb{S}^d)}$$
$$= \left(\frac{\omega_{d-1}}{\omega_d} \int_{-1}^{1} |g(t)|^r (1-t^2)^{\frac{d-2}{2}} \; dt\right)^{1/r} = \left(\frac{\omega_{d-1}}{\omega_d} \int_{0}^{\pi} |g(\cos(\theta))|^r [\sin(\theta)]^{d-1} \; d\theta\right)^{1/r}.$$

With that, and by an application of Holder, Minkowski inequalities and Riesz-Thorin interpolation, one gets, for $\frac{1}{p} = \frac{1}{r} + \frac{1}{q} - 1$,

$$\|f * g\|_{L^p(\mathbb{S}^d)} \le \|f\|_{L^q(\mathbb{S}^d)} \|g\|_{L^r_w([-1,1])}. \tag{4}$$

For more details see [DaXu, Chapter 2].

On $\mathbb{S}^d$ (with obvious metric) we denote the Laplace-Beltrami operator by $\triangle$. With the inner product $\langle f, g\rangle_{\mathbb{S}^d} := \frac{1}{\omega_d} \int_{\mathbb{S}^d} f(x)\overline{g(x)} d\sigma(x)$, the eigenfunctions (spherical harmonics) of $-\triangle$ form an orthonormal basis for $L^2(\mathbb{S}^d)$, in particular

$$f(x) = \sum_{n=0}^{\infty} \mathrm{proj}_n f(x),$$

where the series converges in L^2. Here proj_n is the projection onto $E_{n(n+d-1)}$, the subspace of $L^2(\mathbb{S}^d)$ spanned by the eigenfunctions with eigenvalue $n(n+d-1)$. Recall that (see, e.g., [DaXu, Section 1.2]), these projections are given by a spherical convolution of f:

$$\mathrm{proj}_n f(x) = f * Z_n(x) = \frac{1}{\omega_d} \int_{\mathbb{S}^d} f(y) Z_n(\langle x, y\rangle)\, d\sigma(y), \;\; x \in \mathbb{S}^d. \tag{5}$$

Here, Z_n's are the zonal harmonics:

$$Z_n(\tau) := \left(\tfrac{2n}{d-1} + 1\right) \tfrac{\Gamma(d/2)\Gamma(d+n-1)}{\Gamma(d)\Gamma(n+d/2)} P_n^{(\frac{d-2}{2}, \frac{d-2}{2})}(\tau), \;\; \tau \in [-1,1], \tag{6}$$

where $P_n^{(\alpha,\beta)}$ denotes the Jacobi polynomial. To prove the theorems on the sphere, we need to study zonal harmonics in some detail. In particular, we need to understand the growth and oscillation of the Jacobi polynomials:

Lemma 2.1. *For $d \geq 2$ the have*

$$Z_n(\cos(\theta)) = b_n^+(\theta)e^{in\theta} + b_n^-(\theta)e^{-in\theta} + E_1(\theta, n), \quad \textit{where}$$

$$|b_n^\pm(\theta)| \lesssim \frac{n^{d-1}}{\langle n\theta\rangle^{\frac{d-1}{2}}}, \quad |b_n^\pm(\theta) - b_{n-1}^\pm(\theta)| \lesssim \frac{n^{d-2}}{\langle n\theta\rangle^{\frac{d-1}{2}}}, \quad \textit{and} \quad |E_1(\theta, n)| \lesssim n^{d-3}. \tag{7}$$

Similarly, we find that the zonal spherical harmonic of degree n satisfies

$$Y_n(\theta) = \mathfrak{b}_n^+(\theta)e^{in\theta} + \mathfrak{b}_n^-(\theta)e^{-in\theta} + E_2(\theta, n), \quad \textit{where}$$

$$|\mathfrak{b}_n^\pm(\theta)| \lesssim \frac{n^{\frac{d-1}{2}}}{\langle n\theta\rangle^{\frac{d-1}{2}}}, \quad |\mathfrak{b}_n^\pm(\theta) - \mathfrak{b}_{n-1}^\pm(\theta)| \lesssim \frac{n^{\frac{d-3}{2}}}{\langle n\theta\rangle^{\frac{d-1}{2}}}, \quad \textit{and} \quad |E_2(\theta, n)| \lesssim n^{\frac{d-5}{2}}.$$

Lastly, let $d = 2$ and fix $k \in \mathbb{Z}$. For $n \gg |k|$, we have the expansion

$$Y_n^k(\theta) = \mathfrak{b}_{n,k}^+(\theta)e^{in\theta + ik\phi} + \mathfrak{b}_{n,k}^-(\theta)e^{-in\theta + ik\phi} + E_{2,k}(\theta, n; k), \quad \textit{where}$$

$$|\mathfrak{b}_{n,k}^\pm(\theta)| \lesssim \frac{n^{\frac{1}{2}}}{\langle n\theta\rangle^{\frac{1}{2}}}, \quad |\mathfrak{b}_{n,k}^\pm(\theta) - \mathfrak{b}_{n-1,k}^\pm(\theta)| \lesssim \frac{1}{n^{\frac{1}{2}}\langle n\theta\rangle^{\frac{1}{2}}}, \quad \textit{and} \quad |E_{2,k}(\theta, n)| \lesssim \frac{1}{n^{\frac{1}{2}}\langle n\theta\rangle^{\frac{1}{2}}}.$$

Proof. For all of the above we first note that we may reduce, by symmetry, to considering $[0, \pi/2]$. Equation (6) now allows us to write

$$Z_n(\cos\theta) = c_d(n)P_n^{(\frac{d-2}{2}, \frac{d-2}{2})}(\cos\theta),$$

where Stirling's approximation,

$$n! = \sqrt{2\pi n}\left(\tfrac{n}{e}\right)^n\left(1 + O\left(\tfrac{1}{n}\right)\right), \tag{8}$$

yields $c_d(n) = n^{\frac{d}{2}}\left(1 + O\left(\frac{1}{n}\right)\right)$. The claims of (7) will then follow by establishing

$$P_n^{(\frac{d-2}{2}, \frac{d-2}{2})}(\cos\theta) = \tilde{b}_n^+(\theta)e^{in\theta} + \tilde{b}_n^-(\theta)e^{-in\theta} + \tilde{E}_1(\theta, n), \quad \text{where} \tag{9}$$

$$|\tilde{b}_n^\pm(\theta)| \lesssim \frac{n^{\frac{d-2}{2}}}{\langle n\theta\rangle^{\frac{d-1}{2}}}, \quad |\tilde{b}_n^\pm(\theta) - \tilde{b}_{n-1}^\pm(\theta)| \lesssim \frac{n^{\frac{d-4}{2}}}{\langle n\theta\rangle^{\frac{d-1}{2}}}, \quad \text{and} \quad |\tilde{E}_1(\theta, n)| \lesssim n^{\frac{d-6}{2}}. \tag{10}$$

In that direction, we note that by [BaGa, Theorem 1.2] we have, for $\theta \in [0, \pi/2]$,

$$P_n^{(\frac{d-2}{2}, \frac{d-2}{2})}(\cos\theta) = \frac{\Gamma(n + d/2)}{n!}\left[\alpha_d(\theta)\frac{J_{\frac{d-2}{2}}(M\theta)}{(M\theta)^{\frac{d-2}{2}}} + \beta_d(\theta)\frac{J_{d/2}(M\theta)}{(M\theta)^{d/2}} + O\left(\tfrac{\theta^2}{n^2}\right)\right] \tag{11}$$

where $M = n + \frac{d-1}{2}$, $|\alpha_d(\theta)| \lesssim 1, |\beta_d(\theta)| \lesssim \theta^2$, and J_γ is the Bessel function of the first kind of order γ. Noting that Stirling's approximation again gives

$$\frac{\Gamma(n + d/2)}{n!} = n^{\frac{d-2}{2}}\left(1 + O\left(\tfrac{1}{n}\right)\right), \tag{12}$$

we reduce to finding suitable bounds on $J_{\frac{d-2}{2}}(M\theta)$ and $J_{\frac{d}{2}}(M\theta)$. This, however, follows by clasically relating Bessel functions to Fourier transforms of surface measures associated to spheres. Let σ_{d-1} be the surface measure of the $d-1$ dimensional unit sphere, $d \geq 2$. We have

$$(2\pi)^{\frac{d}{2}} \frac{J_{\frac{d-2}{2}}(|\xi|)}{|\xi|^{\frac{d-2}{2}}} = \widehat{\sigma_{d-1}}(\xi) = \int e^{-ix\cdot\xi}\, d\sigma_{d-1}(x) = w_{d-1}^{+}(|\xi|)e^{i|\xi|} + w_{d-1}^{-}(|\xi|)e^{-i|\xi|},$$

where

$$|w_{d-1}^{\pm}(r)| \lesssim \langle r\rangle^{-\frac{d-1}{2}}, \qquad \text{and} \qquad |\partial_\theta w_{d-1}^{\pm}(r)| \lesssim \langle r\rangle^{-\frac{d+1}{2}}.$$

Using these and (12) on the right hand of (11), we obtain the representation given in (9) and (10). The bound for $Y_n(\theta)$ follows from the above by noting that

$$\begin{aligned} Y_n(\theta) &:= \sqrt{\tfrac{(2n+d-1)\Gamma(n+d-1)\Gamma(n+1)}{2^{d-1}\Gamma(n+1+\frac{d-2}{2})^2}}\, P_n^{\frac{d-2}{2},\frac{d-2}{2}}(\cos(\theta)) \\ &= n^{1/2}\left(1 + O\left(\tfrac{1}{n}\right)\right) P_n^{\frac{d-2}{2},\frac{d-2}{2}}(\cos(\theta)). \end{aligned} \tag{13}$$

The final claim of the lemma follows from the prior work and an expansion of [OlF, Chapter 12, Section 13]:

$$Y^{n,k}(\theta,\phi) = (-1)^k e^{ik\phi} n^{1/2}\big(\frac{\theta}{\sin\theta}\big)^{1/2}\big(J_{|k|}((n+1/2)\theta) + O_{|k|}(\frac{1}{n})\,\mathrm{env}\, J_{|k|}((n+1/2)\theta)\big),$$

where env denotes the envelope of $J_{|k|}$ and $\theta \in (0, \pi/2)$. The asymptotics then follow exactly as above, where the weaker error bound is due to the error only being in terms of $J_{|k|}$. □

Our next aim is to introduce Besov spaces by utilizing the following proposition and to obtain related dimension bounds for the graphs of continuous functions on the spheres.

Proposition 2.2 ([NPW, NPW2])**.** *Let $a \in C^\infty([0,\infty))$ satisfy supp $a \subset [1/2, 2]$, $|a(t)| > c > 0$ for $t \in [3/5, 5/3]$, and $a(t) + a(2t) = 1$ if $t \in [1/2, 1]$, and define*

$$\Phi_0(t) := Z_0(t), \quad \Phi_j(t) := \sum_{n=0}^{\infty} a(2^{-j+1}n)Z_n(t), \quad t \in [-1,1], \;\; j \geq 1, \quad \textit{and}$$

$$\mathcal{K}_j(\cos\theta) := \sum_{i=0}^{j} \Phi_i(\cos\theta).$$

Then for all $j, k \geq 0$

$$|\Phi_j(\cos\theta)| + |\mathcal{K}_j(\cos\theta)| \lesssim_{d,k} \frac{2^{jd}}{(1+2^j\theta)^k}, \tag{14}$$

and if $f \in L^p$ for $1 \leq p < \infty$ and $f \in C^0$ when $p = \infty$, then

$$f * \mathcal{K}_j \overset{j\to\infty}{\to} f \text{ in } L^p.$$

We define the smooth cut-off Besov spaces, [NPW], as:

$$\|f\|_{B^{\alpha}_{p,\infty}} := \sup_{j\geq 0} 2^{j\alpha}\|\mathbb{P}_{2^j} f\|_{L^p} := \sup_{j\geq 0} 2^{j\alpha}\|f * \Phi_j\|_{L^p}.$$

With these preliminaries out of the way, using (14) it is then standard exercise in dyadic decomposition to find the following characterization of $C^{\alpha}(\mathbb{S}^d)$:

Theorem 2.3. *For $0 < \alpha < 1$, if $f \in C^{\alpha}(\mathbb{S}^d)$ then $\|f * \mathcal{K}_j - f\|_{\infty} \lesssim 2^{-j\alpha}$. Conversely, $\|f * \mathcal{K}_j - f\|_{\infty} \lesssim 2^{-j\alpha}$ implies $f \in C^{\alpha}(\mathbb{S}^d)$.*

This theorem leads to

Theorem 2.4. *The spaces $C^{\alpha}(\mathbb{S}^d)$ and $B^{\alpha}_{\infty,\infty}(\mathbb{S}^d)$ coincide for $\alpha \in (0, 1)$. In particular, if $f \in B^{\alpha}_{\infty,\infty}(\mathbb{S}^d)$, the fractal dimension of its graph must be bounded above by $(d + 1) - \alpha$.*

For proofs of these theorems using Césaro means we refer the reader to [Hu]– the proof of the above follows in the same manner, utilizing (14). In order to obtain a version of the Deliu-Jawerth theorem for $\mathbb{S}^d$ we will need to introduce the spherical analogue of translation.

Definition 2.5. [DaXu, Prop 2.1.5] *Let $\theta \in [0, \pi]$ and $f \in L^2(\mathbb{S}^d)$. For $x \in \mathbb{S}^d$, let $\mathbb{S}^{\perp}_x := \{y \in \mathbb{S}^d : \langle x, y\rangle = 0\}$ be the equator in $\mathbb{S}^d$ with respect to x. The average shift operator T_θ is defined as*

$$T_\theta f(x) := \frac{1}{\omega_{d-1}} \int_{\mathbb{S}^{\perp}_x} f(x \cos\theta + u \sin\theta) d\sigma(u).$$

For $g : [-1, 1] \to \mathbb{R}$, we have

$$f * g(x) = \frac{\omega_{d-1}}{\omega_d} \int_0^{\pi} T_\theta f(x) g(\cos\theta)(\sin\theta)^{d-1}\, d\theta.$$

$T_\theta f(x)$, as defined, is really an average of the values of f evaluated at all points y such that the geodesic distance $d(x, y) = \theta$. Consequently, it is also referred to in literature as the average shift operator on $\mathbb{S}^d$. It has all the properties we desire from a translation operator; in addition to the convolution identity above, we have

Lemma 2.6. [DaXu, Pg. 32] *For $f \in L^p(\mathbb{S}^d), 1 \leq p < \infty$ or $f \in C^0(\mathbb{S}^d)$ for $p = \infty$,*

$$\|T_\theta f\|_{L^p} \leq \|f\|_{L^p}, \qquad \lim_{\theta\to 0+} \|T_\theta f - f\|_{L^p} = 0.$$

Moreover, T_θ respects the symmetries of $\mathbb{S}^d$ by definition. We also need the following counting lemma.

Lemma 2.7. *Let $\theta \in [0, \pi]$. For $f : \mathbb{S}^d \to \mathbb{R}$ continuous function,*

$$\|T_\theta f - f\|_{L^1(\mathbb{S}^d)} \lesssim \theta^{d+1}\mathcal{N}(E, \theta)$$

where E is the graph of f, and $\mathcal{N}(E,\theta)$ is the minimum number of balls of radius $\theta > 0$ necessary to cover E.

Proof. Let us denote the cap of radius θ centered at $x \in \mathbb{S}^d$ by $\Theta = \Theta(x)$. If $y \in \Theta$ then we may bound

$$|f(x) - f(y)| \lesssim \theta N_\theta^x,$$

where N_θ^x is the minimum number of balls of radius θ required to cover the graph of f above Θ. This is independent of y, so this must also hold for the average over $y \in \partial\Theta$, and hence

$$|f(x) - T_\theta f(x)| \lesssim \theta N_\theta^x.$$

We now decompose $\mathbb{S}^d$ into a finite number of $\Theta(x_i)$ caps, each centered at x_i. Under this decomposition we find

$$\begin{aligned}\|f - T_\theta f\|_{L^1} \lesssim \int \theta N_\theta^x \, d\sigma(x) &\lesssim \sum_{\Theta(x_i)} \theta \int_{\Theta(x_i)} N_\theta^x \, d\sigma(x) \\ &\lesssim \sum_{\Theta(x_i)} \theta \int_{\Theta(x_i)} N_{5\theta}^{x_i} \, d\sigma(x) \lesssim \theta^{d+1} \sum_{\Theta(x_i)} N_\theta^{x_i} \lesssim \theta^{d+1} N(E,\theta),\end{aligned}$$

as desired. □

We are now ready to prove a key result that gives lower bounds for graph dimension. The original theorem was proven by Deliu and Jawerth in [DeJa] for continuous functions $\mathbb{T} \to \mathbb{R}$. The proof we present below extends the ideas used in the proof given in [ErTz3, Theorem 2.24] to continuous functions $\mathbb{S}^d \to \mathbb{R}$, after translating the integral over $\mathbb{S}^d$ into the integral involving T_θ over the interval $[0,\pi]$.

Theorem 2.8. *For a continuous function $f : \mathbb{S}^d \to \mathbb{R}$, the graph E of f has fractal dimension $D \geq (d+1) - s$ provided that $f \notin \bigcup_{\epsilon>0} B_{1,\infty}^{s+\epsilon}(\mathbb{S}^d)$.*

Proof. It suffices to prove if f is continuous and $\sup_{j\in\mathbb{N}} 2^{js}\|f * \Phi_j\|_{L^1} = \infty$ for some $0 < s < 1$, then the graph E of f has dimension $D \geq (d+1) - s$.

Recalling Definition 2.5 on the operator T_θ, we rewrite $\|\mathbb{P}_{2^j} f\|_{L^1}$ (for $j \geq 1$):

$$\begin{aligned}\|\mathbb{P}_{2^j} f\|_{L^1_x} &\sim \left\| \int_{\mathbb{S}^d} \Phi_j(\langle x,y\rangle)[f(y) - f(x)] \, d\sigma(y) \right\|_{L^1_x} \\ &\sim \left\| \int_0^\pi \Phi_j(\cos\theta)[T_\theta f(x) - f(x)](\sin(\theta))^{d-1} \, d\theta \right\|_{L^1_x}.\end{aligned}$$

Let us consider the inner integral, and recall (14). Dyadically splitting θ and choosing $k > d + s$, we find by Lemma 2.7:

$$\|\mathbb{P}_{2^j} f\|_{L^1_x} \lesssim \int_0^\pi |\Phi_j(\cos\theta)| \|T_\theta f - f\|_{L^1_x} \theta^{d-1} \, d\theta$$

$$\lesssim \int_0^\pi 2^{dj}(1+2^j\theta)^{-k}\theta^{2d}\mathcal{N}(E,\theta)\,d\theta$$
$$\lesssim \sum_{\ell=0}^\infty \int_{2^{-\ell-1}<\theta\le 2^{-\ell}} 2^{dj-2d\ell}(1+2^{j-\ell})^{-k}\mathcal{N}(E,2^{-\ell})\,d\theta$$
$$\lesssim \sum_{\ell=0}^\infty 2^{dj-2d\ell-\ell}(1+2^{j-\ell})^{-k}\mathcal{N}(E,2^{-\ell}).$$

Multiplying by 2^{sj} and taking the supremum in j, we find

$$\|f\|_{B^s_{1,\infty}} \lesssim \sum_{\ell=0}^\infty \sup_{j\ge 0} 2^{dj-2d\ell-\ell+sj}(1+2^{j-\ell})^{-k}\mathcal{N}(E,2^{-\ell}) \lesssim \sum_{\ell=0}^\infty 2^{-\ell(d+1-s)}\mathcal{N}(E,2^{-\ell}),$$

as the supremum is attained for $j=\ell$ (since $k>d+s$). As this must diverge, we conclude that $\mathcal{N}(E,2^{-\ell}) \gtrsim 2^{\ell(d+1-s)}/\ell^2$ infinitely often, and hence the claim follows. □

3. Talbot Effect On the Sphere

In this section we will prove theorems displaying fractal behavior of solutions to the linear Schrödinger equation (1). The propagator of the Schrödinger equation (1) on $\mathbb{S}^d$ is given by

$$e^{it\triangle}f = \sum_n e^{itn(n+d-1)}\mathrm{proj}_n f, \tag{15}$$

where $\mathrm{proj}_n f$ is defined in (5) as the projection to eigenspace corresponding to the eigenvalue $n(n+d-1)$.

Remark 3.1. *For dyadic N, we define the sharp cut-off Littlewood-Paley projection operators P_N by*

$$P_N(f) = \sum_{N\le|n|<2N} proj_n f = f * \Big(\sum_{N\le|n|<2N} Z_n\Big).$$

It suffices to obtain upper bounds for these projections as the projections with smooth cut-offs are uniformly bounded in L^p spaces.

In addition to the results we presented in the previous section, we will make use of $L^4(\mathbb{S}^d\times[0,2\pi])$ estimates of [BGZ, BGZ2]:

$$\|e^{it\triangle}f(x,t)\|_{L^4_{x,t}(\mathbb{S}^d\times[0,2\pi])} \lesssim_\varepsilon \begin{cases} \|f\|_{H^{\frac{1}{8}+\varepsilon}(\mathbb{S}^2)} & d=2 \\ \|f\|_{H^{\frac{d-2}{4}+\varepsilon}(\mathbb{S}^d)} & d\ge 3, \end{cases} \tag{16}$$

where

$$\|f\|_{H^s(\mathbb{S}^d)} := \sqrt{\sum_{N \text{ dyadic}} N^{2s}\|P_N f\|^2_{L^2(\mathbb{S}^d)}}.$$

Up to ε, the above estimates are optimal L^4 bounds, see [BGZ2] and Remark 5.2. It's worth noting that what is known on $\mathbb{S}^d$ is much weaker than on $\mathbb{T}^d$. Specifically, [BoDe] obtained the full gamut of Strichartz estimates for $\mathbb{T}^d$:

$$\|e^{it\triangle_{\mathbb{T}^d}} f(x,t))\|_{L^p_{x,t}(\mathbb{T}^d\times[0,2\pi])} \lesssim_\varepsilon \begin{cases} \|f\|_{H^{\frac{d}{2}-\frac{d+2}{p}+\varepsilon}(\mathbb{T}^d)} & p \geq \frac{2(d+2)}{d} \\ \|f\|_{H^\varepsilon(\mathbb{T}^d)} & 2 < p < \frac{2(d+2)}{d}. \end{cases} \tag{17}$$

Unlike (16), these estimates always include a region of $p > 2$ for which there is only an ε-derivative loss. In Lemma 5.1 below, we obtain improved L^4 bound with only ε loss when restricted to zonal spherical harmonics.

The next lemma is standard. Specifically, it demonstrates the expected square root cancellation for weighted Weyl sums with decaying weights, see [ErTz3, ErSh, Hu].

Lemma 3.2. *Consider the exponential sum $\sum_{n=N}^u e^{in^2t+inx}a^n b_n$ for $x \in \mathbb{R}$. Assume $a \in [0,1]$ and for each $n \in \mathbb{N}$, b_n satisfies*

$$|b_n| \lesssim \frac{1}{n^p}, \qquad |b_n - b_{n-1}| \lesssim \frac{1}{n^{p+1}},$$

for some $p \in \mathbb{R}$. It follows that for almost every t, we have for all $N \in \mathbb{N}$:

$$\sup_{N\leq u\leq 2N} \sup_{x\in\mathbb{R}} \left| \sum_{n=N}^{u} e^{in^2t+inx}\, a^n\, b_n \right| \lesssim_t N^{1/2-p}. \tag{18}$$

Remark 3.3. *The above factor of a^n is not important for many of our applications, but is included because it makes the presentation of applications to functions supported on Gaussian beams simpler.*

With this out of the way, we're ready to prove our first theorem.

Proof of Theorem 1.1. By Theorem 2.4, it suffices to prove that for almost all t, $u(t,\cdot)$ belongs to $B^{\gamma-}_{\infty,\infty}$. By Remark 3.1, it suffices to prove that

$$\|P_N(u(t,x))\|_{L^\infty_x(\mathbb{S}^d)} \lesssim N^{-\gamma+}.$$

We write $P_N(u(x,t)) = f_N * H_N(x,t)$, where

$$H_N(\cos\theta, t) := \sum_{N<n\leq 2N} e^{itn(n+d-1)} Z_n(\cos\theta) \tag{19}$$

and the convolution is the spherical convolution. Using (4), we have (for $\frac{1}{p}+\frac{1}{q}=1$):

$$\|P_N(u(t,x))\|_{L^\infty_x(\mathbb{S}^d)} \lesssim \|f_N\|_{L^p(\mathbb{S}^d)} \|H_N(\cdot,t)\|_{L^q_w([-1,1])}, \tag{20}$$

where $\|\cdot\|_{L^q_w([-1,1])}$ is defined by (3).

By Lemma 2.1, we have (by symmetry, $\theta \in [0, \frac{\pi}{2}]$)

$$Z_n(\cos(\theta)) = b_n^+(\theta)e^{in\theta} + b_n^-(\theta)e^{-in\theta} + E(\theta, n),$$

where $b_n^\pm$ satisfies (7) and hence also the conditions of Lemma 3.2 with $N < n \leq 2N$, $a = 1$ and p depending on θ, N is

$$p(\theta, N) = \begin{cases} -\frac{d-1}{2} & \theta \in [\frac{1}{N}, \frac{\pi}{2}] \\ -(d-1) & \theta \in [0, \frac{1}{N}]. \end{cases} \tag{21}$$

Now, by incorporating the phase factors from above into the exponential and using the trivial estimate on the error, we see that Lemma 3.2 yields, for almost all t:

$$|H_N(\cos\theta, t)| \lesssim_t \frac{N^{d-\frac{1}{2}}}{\langle N\theta\rangle^{\frac{d-1}{2}}}.$$

Using this we have

$$\begin{aligned} \|H_N(\cdot, t)\|^q_{L^q_w([-1,1])} &\lesssim \int_0^{\frac{\pi}{2}} |H_N(\cos\theta, t)|^q \sin^{d-1}\theta \, d\theta \\ &\lesssim_t \int_0^{\frac{\pi}{2}} \frac{N^{q(d-\frac{1}{2})}}{\langle N\theta\rangle^{q\frac{d-1}{2}}} \theta^{d-1}\, d\theta \lesssim N^{q(d-\frac{1}{2})-d} + \begin{cases} N^{\frac{dq}{2}+} & \text{if } 2 + \frac{2}{d-1} \geq q \\ N^{q(d-\frac{1}{2})-d} & \text{if } q > 2 + \frac{2}{d-1}. \end{cases} \end{aligned}$$

Together with (20) and the hypothesis $\|f_N\|_{L^p} \lesssim N^{-\left(\frac{d}{2}+s\right)}$, we then have

$$\|P_n(u(\cdot, t))\|_{L^\infty_x(\mathbb{S}^d)} \lesssim \|f_N\|_{L^p} \|H_N\|_{L^q_w([-1,1])} \lesssim \max\left(N^{-s+}, N^{d\left(\frac{1}{p}-\frac{1}{2}\right)-\frac{1}{2}-s+}\right).$$

This then gives $u(t, x) \in C^{\gamma-}$ for almost all t and $\gamma = \min\{s, s + \frac{d+1}{2} - \frac{d}{p}, 1\}$. It also implies the upper bound of $(d+1) - \gamma$ on the fractal dimension of graph of $u(t, x)$ for almost every t.

For the lower bound, following the arguments of [ErSh], we see that the assumption on $\|f_N\|_{L^p_x}$ and Sobolev embedding imply ($d = 2$)

$$\|f_N\|_{L^2_x} \lesssim N^{\frac{2}{p}-1} \|f_N\|_{L^p_x} \lesssim N^{\frac{2}{p}-2-s},$$

so that we find by (16)

$$\|\langle\nabla\rangle^{-(\frac{1}{8}+\frac{2}{p}-2-s)-} u\|_{L^4_{x,t}} \lesssim \|\langle\nabla\rangle^{-(\frac{2}{p}-2-s)-} f\|_{L^2_x} \lesssim 1.$$

Thus, for almost every time, t, we find

$$\|P_N(u)\|_{L^4_x} \lesssim_t N^{(\frac{1}{8}+\frac{2}{p}-2-s)+}.$$

We now assume that $f \notin H^{s+2-\frac{2}{p}}$, so that interpolation between the L^1 and the L^4 bound gives

$$\sup_N N^\gamma \|P_N(u)\|_{L^1_x} = \infty,$$

for $\gamma > 2 - \frac{2}{p} + \frac{1}{4} + s$. It follows by Theorem 2.8 that the fractal dimension of the graph of u is bounded below by $\max\left(\frac{3}{4} + \frac{2}{p} - s, 2\right)$ for almost all t. □

Remark 3.4. *The upper bound above is best when $p = \frac{2d}{d+1} \to 2$ as $d \to \infty$. At this level, the bound for H_N begins to match the bound for the torus, see Theorem 6.1.*

3.1. **Specific Expansions on $\mathbb{S}^2$.** In this subsection we explore dimension bounds for functions supported on specific spherical harmonics in the case that $d = 2$. In particular, we will focus primarily on functions supported on the Zonal harmonics and Gaussian beams, which are of interest because they are the extremal harmonics in some sense (explored further in Appendix 5). However, we also demonstrate results for more general expansions (of the same form as Zonal and Guassian Beams) that follow from the same methods.

When $d = 2$, we have an explicit formula for the spherical harmonics of degree n (see for example [DaXu, Section 1.6]):

Theorem 3.5. *On $\mathbb{S}^2$, let θ, ϕ denote the azimuthal and polar angles respectively in spherical coordinates ($0 \le \theta \le \pi, 0 \le \phi < 2\pi$). For $-n \le k \le n$, Define*

$$Y_n^k(\theta, \phi) := \sqrt{\tfrac{(2n+1)(n-k)!}{(n+k)!}} P_n^k(\cos\theta) e^{ik\phi}. \tag{22}$$

Then $\{Y_n^k : n \in \mathbb{N}_0, -n \le k \le n\}$ forms an orthonormal basis of $L^2(\mathbb{S}^2)$.

By inductive construction, one can also obtain basis elements for $L^2(\mathbb{S}^d)$ for $d > 2$, see [Hu].

The Zonal harmonic of degree n, denoted[3] Y_n^0, has an explicit form given by (13). In particular, we find

$$Y_n^0(\theta, \phi) = \sqrt{2n+1} P_n(\cos\theta), \tag{23}$$

where P_n is now the Legendre polynomial of degree n. Gaussian beams, denoted $Y_n^{\pm n}$, have a similarly nice expression of the form

$$Y_n^{\pm n}(\theta, \phi) = \sqrt{(2n+1)\binom{2n}{n}} \frac{(\mp 1)^n}{2^n} \sin^n(\theta). \tag{24}$$

[3]We comment that when $d > 2$ these are denoted Y_n. When $d = 2$ the spherical harmonics are usually denoted as some variant of Y_n^k for $|k| \le n$. The case that $k = 0$ corresponds to the Zonal harmonics on $\mathbb{S}^2$.

Proof of Theorem 1.3. We will show for almost all t and N dyadic

$$\|P_N(u(\cdot,t))\|_{L^\infty} \lesssim N^{-(p-1)+}.$$

We use again Lemma 2.1 and summation-by-parts as in the proof of Theorem 1.1, noting that $b_n = a_n \mathfrak{b}_n^{\pm}$ and $a = 1$ satisfies the hypothesis of Lemma 3.2 with

$$|b_n| \lesssim \frac{n^{\frac{1}{2}-p}}{\langle n\theta\rangle^{\frac{1}{2}}}$$

It follows that

$$\|P_n(u(\cdot,t))\|_{L^\infty_x} \lesssim \sup_\theta \frac{N^{1-p}}{\langle N\theta\rangle^{\frac{1}{2}}} \lesssim N^{1-p},$$

establishing the upper bound claimed.

In order to establish a lower bound we must refine the Strichartz estimate (16). This is done in the appendix, Lemma 5.1. In particular, we find

$$\|u\|_{L^4_{x,t}} \lesssim \|f\|_{H^\varepsilon},$$

for any $\varepsilon > 0$. We now assume that $f \notin H^{p-\frac{1}{2}+}(\mathbb{S}^2)$. Interpolation as in Theorem 1.1 with the improved L^4 bound forces the dimension to be bounded below by

$$\max\left(3 - \left(p - \tfrac{1}{2}\right), 2\right) = \max\left(\tfrac{7}{2} - p, 2\right). \qquad \square$$

Remark 3.6. *The above lower bound is only meaningful when $1 \le p \le \frac{3}{2}$.*

We can also easily extend the above upper bound to zonal harmonics on $\mathbb{S}^d$ by using the full asymptotics of Lemma 2.1. In particular, we find Corollary 1.4 holds, whose proof follows in the exact same way as above.

One doesn't have to stop at just Zonal harmonics. Inded, Lemma 2.1 yields asymptotics for Y_n^k, under the assumption that k is fixed. Using these asymptotics we may establish dimension bounds similar to Theorem 1.3.

Lemma 3.7. *Let k be fixed and $f(\theta,\phi) = \sum_{n\ge k}^{\infty} a_n Y_n^k(\theta,\phi)$. If for some $1 < p < 2$ we have*

$$|a_n| \lesssim \tfrac{1}{n^p}, \text{ and } \quad |a_n - a_{n-1}| \lesssim \tfrac{1}{n^{p+1}}$$

for all $n \in \mathbb{N}\cup\{0\}$, then for almost all t, $u(x,t) \in C^{(p-1)-}$, and hence $\dim_t(f) \le 4 - p$. If, in addition, $f \notin H^{p-\frac{1}{2}+}(\mathbb{S}^2)$, then we also find $\dim_t(f) \ge \max\left(\frac{15}{4} - p, 2\right)$.

Proof. The upper bound follows immedietly from Lemma 2.1, as in Theorem 1.3. The lower bound similarly follows from interpolation using the Strichartz estimate (16). $\square$

Instead of fixing k, we also establish bounds when $k=-n$, which corresponds to the case of Guassian beams.

Theorem 3.8. *Let* $f(x)=f(\theta,\phi)=\sum_{n=0}^{\infty}a_nY_n^{-n}(\theta,\phi)$*. If for some* $\frac{3}{4}<p<\frac{5}{4}$ *we have,*

$$|a_n|\lesssim\frac{1}{n^p},\ \ \textit{and}\quad |a_n-a_{n-1}|\lesssim\frac{1}{n^{p+1}}$$

for all $n\in\mathbb{N}\cup\{0\}$*, then for almost all* t *and* $\dim_t(f)\le\frac{15}{4}-p$*.*

If, in addition, $f\notin H^{p-1/2+}$*, then* $\dim_t(f)\ge\max(\frac{13}{4}-p,2)$*.*

Proof. Recall that (referencing Theorem 3.5)

$$\begin{aligned}Y_n^{-n}(\theta,\phi)&=\sqrt{(2n+1)(2n)!}P_n^{-n}(\cos\theta)e^{-in\phi}\\&=\sqrt{\tfrac{2n+1}{(2n)!}}(-1)^nP_n^n(\cos\theta)e^{-in\phi}\end{aligned}$$

Furthermore, using the closed form formula for associated Legendre polynomial [Fö], we have:

$$P_n^n(\cos\theta)=(-1)^n\,2^n\,n!\tbinom{n-1/2}{n}(1-\cos^2\theta)^{n/2}=(-1)^n\,\frac{n!}{2^n}\tbinom{2n}{n}(\sin\theta)^n$$

We can now rewrite $u(x,t)$ using this above information:

$$\begin{aligned}u(\theta,\phi,t)&=\sum_{n=0}^{\infty}e^{in(n+1)t}\,a_nY_n^{-n}(\theta,\phi)\\&=\sum_{n=0}^{\infty}a_n\,e^{in(n+1)t}\sqrt{\tfrac{2n+1}{(2n)!}}P_n^n(\cos\theta)e^{-in\phi}\\&=\sum_{n=0}^{\infty}e^{in(n+1)t}e^{-in\phi}\sin^n(\theta)\,a_n\sqrt{2n+1}\sqrt{\tbinom{2n}{n}}\,\frac{1}{2^n}.\end{aligned}$$

In order to obtain an upper bound on the fractal dimension of the graph of $u(x,t)$, we need to estimate

$$\|P_N(u(\cdot,t))\|_{L^\infty_x}=\sup_{(\theta,\phi)\in\mathbb{S}^2}\left|\sum_{N<n\le 2N}e^{in(n+1)t}e^{-in\phi}\sin^n(\theta)\ a_n\sqrt{2n+1}\sqrt{\tbinom{2n}{n}}\,\frac{1}{2^n}\right|$$

for N dyadic.

Once again, Lemma 3.2 can be used for fixed θ with $a=\sin\theta$ and $b_n=a_n\sqrt{2n+1}\sqrt{\binom{2n}{n}}\,\frac{1}{2^n}$. We note that by Stirling's approximation (8), we find

$$\sqrt{\tbinom{2n}{n}}=\sqrt{\tfrac{2^{2n}}{\sqrt{n}}}\ \big(c+O(\tfrac{1}{n})\big).$$

Consequently

$$|b_n|\lesssim\left|a_n\sqrt{2n+1}\sqrt{\tfrac{2^{2n}}{\sqrt{n}}}\,\frac{1}{2^n}\right|\lesssim|a_nn^{1/4}|.$$

By the assumption on a_n, we obtain for b_n:

$$|b_n| \lesssim \frac{1}{n^{p-1/4}}; \qquad |b_n - b_{n-1}| \lesssim \frac{1}{n^{p+3/4}}. \tag{25}$$

This leads to the following for almost all t:

$$\sup_{\substack{\theta\in[0,\pi/2]\\ \phi\in[0,2\pi]}} \left| \sum_{N<n\leq 2N} e^{in(n+1)t} e^{-in\phi} \sin^n(\theta)\; a_n \sqrt{2n+1} \sqrt{\binom{2n}{n}} \frac{1}{2^n} \right| \lesssim \frac{N^{1/2+}}{N^{p-1/4}} = N^{-(p-3/4)+}.$$

Given the assumption on p, this implies the solution $u(x,t)$ is in $C^{(p-3/4)-}$ for almost all t, thus the fractal dimension of its graph is bounded above by $3 - \left(p - \frac{3}{4}\right) = \frac{15}{4} - p$.

As there is no hope of improving the Strichartz estimate for these harmonics, the lower bound follows by the same interpolation argument as Theorem 1.1 and Lemma 3.7. □

Remark 3.9. *Similar statements are available for combinations of specific harmonics, but depend greatly on the specific form of the eigenfunction. Because of this it seems difficult to obtain a result akin to Theorem 6.4 for* $\mathbb{S}^2$.

Remark 3.10. *The above corollary isn't unique to* $Y_n^{\pm n}$ *– a similar statement will hold in the exact same way for any function supported on* $\{Y_n^{g(n)}\}_n$ *with* $\left|\frac{g(n)}{n}\right| \to 1$ *as* $n \to \infty$.

We can say more than the above theorem when off of the equator. Specifically, because of the factor of $\sin^n(\theta)$, we find that all functions supported on the Gaussian beams with polynomially growing coefficients is $C^\infty_{x,t}(S_\delta)$ for

$$S_\delta = \left\{(\theta,\phi) \in \mathbb{S}^2 \,:\, \theta \notin \left(\tfrac{\pi}{2} - \delta, \tfrac{\pi}{2} + \delta\right)\right\},$$

and $0 < \delta < \frac{\pi}{2}$. This is recorded in the next corollary.

Corollary 3.11. *Let* $\alpha, \beta \in \mathbb{R}$, $0 < \delta < \frac{\pi}{2}$, *and* $f(x) = \sum_{n=0}^{\infty} a_n Y_n^{-n}(x)$ *for some scalars* a_n *with* $|a_n| \lesssim n^\alpha$. *If* $\mu : \mathbb{R} \mapsto \mathbb{R}$ *satisfies* $|\mu(x)| \lesssim |x|^\beta$ *and*

$$u(x,t) = e^{it\mu(-\triangle)} f = \sum_{n=0}^{\infty} e^{it\mu(n(n+1))} a_n Y_n^{-n}(x),$$

then $u \in C^\infty_{x,t}(S_\delta)$. *It follows that the dimension of* $u(S_\delta, t)$ *is exactly* 2 *for all* t.

Proof. This is fairly trivial. Let $x = (\theta,\phi)$ and $k, \ell \in \mathbb{N}$. We then find

$$\begin{aligned} |\partial_t^k \triangle^\ell u(x,t)| &= \left| \sum_{n=0}^{\infty} e^{it\mu(n(n+1))} \mu(n(n+1))^k n^\ell (n+1)^\ell a_n \cdot Y_n^{-n}(x) \right| \\ &\lesssim \sum_{n=0}^{\infty} |n|^{2k+2\beta+\alpha+\frac{1}{4}} \sin^n\left(\frac{\pi}{2} - \delta\right) < \infty, \end{aligned}$$

uniformly for $x \in S_\delta$. □

4. Nonlinear Smoothing for the Zonal Cubic NLS on $\mathbb{S}^d$

In this section we derive a smoothing statement for the cubic NLS on $\mathbb{S}^d$ restricted to

$$\mathcal{Z}^s(\mathbb{S}^d) := \Big\{ f \in H^s(\mathbb{S}^d) : \, f = \sum_{n=0}^{\infty} a_n Y_n \Big\}.$$

Here $a_n = \widehat{f}(n) = \mathcal{F}(f)(n) = \frac{1}{\omega_d} \int_{S^d} f(x) Y_n(x) \, d\sigma(x)$.

We have Parseval's identity:

$$\sum_{n=0}^{\infty} \widehat{f}(n) \bar{\widehat{g}}(n) = \frac{1}{\omega_d} \int_{\mathbb{S}^d} f(x) \bar{g}(x) d\sigma(x).$$

We also define

$$\kappa(n, n_1, ..., n_j) := \mathcal{F}\big(\prod_{i=1}^{j} Y_i\big)(n) = \frac{1}{\omega_d} \int_{\mathbb{S}^d} Y_n(x) \prod_{i=1}^{j} Y_i(x) \, d\sigma(x). \tag{26}$$

Note that κ is independent of the order of the indices. By [Ga], $\kappa \geq 0$ for any choice of indices, and $\kappa = 0$ if any index is strictly greater than the sum of the others. Finally, by the Parseval's identity above, we have

$$\sum_n \kappa(n, n_1, ..., n_j) \kappa(n, m_1, ..., m_\ell) = \kappa(n_1, ..., n_j, m_1, ..., m_\ell).$$

Remark 4.1. *These facts about κ allows one to perform multilinear estimates in the standard way– that is, by assuming positivity of either the Fourier transforms or the space-time Fourier transforms and pulling absolute values in. The non-negativity specifically guarantees access to Parseval and Plancherel after pulling in the absolute values.*

We consider solutions to

$$\begin{cases} i\partial_t u + \triangle_{\mathbb{S}^d} u \pm |u|^2 u = 0 \\ u(x, 0) = u_0(x) \in \mathcal{Z}^s(\mathbb{S}^d). \end{cases} \tag{27}$$

As a consequence of Lemma 5.1 and (16) we find the following proposition.

Proposition 4.2. *For $s > \frac{d-2}{2}$ the equation* (27) *is locally well-posed with a time of existence $T = T(\|u_0\|_{H^s}) > 0$.*

Before moving on to the statement of the result, we first describe the nature of the result, leaving the historical background of the result to [ErTz3, Mc]. The linear group of the NLS is an isometry on L^2 based spaces, and hence we can not expect the linear group to present any smoothing behaviour, i.e., it cannot be in an higher index L^2-based Sobolev space for any t. By the Duhamel representation, we have

$$u(t, x) - e^{it\triangle_{\mathbb{S}^d}} u_0(x) = \mp \int_0^t e^{i(t-s)\triangle_{\mathbb{S}^d}} |u|^2(s, x) u(s, x) \, ds.$$

One could expect such a statement to possibly hold for the nonlinear part of the evolution, i.e., the right hand side of the formula above. On many periodic domains this, too, fails. This is best seen through the result of [ErTz2], which demonstrates that on $\mathbb{T}$ one has a $u\|u_0\|^2_{L^2_x}$ term sitting in the non-linearity arising from resonances. This automatically precludes extra smoothness of the integral term in the Duhamel representation. The fix to this is to introduce a phase rotation in order to remove this term from the differential equation, modifying the equation to its *Wick reordering*. Indeed, the correct statement is then

$$u - e^{it\left(\triangle_{\mathbb{T}} + \frac{1}{\pi}\|u_0\|_{L^2_x(\mathbb{T})}\right)} u_0 \in C^0\left([0,T), H^{s+\varepsilon}_x(\mathbb{T})\right),$$

for $0 \leq \varepsilon < \min(2s, 1/2)$, [ErTz2], which brings us back to the statement of Theorem 1.5.

Proof of 1.5. The proof of Theorem 1.5 follows from an application of Lemmas 4.11, 4.12, and 4.13 below to the Duhamel representation associated to (40), together with the local well-posedness bound for $0 \leq t < T$. □

Remark 4.3. *In particular, γ is a real function depending on the solution u, so that the above theorem states that the solution, up to a phase rotation of the initial data, is in a smoother space than the initial data.*

As an application, we prove dimension bounds for the nonlinear evolution.

Proof of 1.6. We first write

$$e^{-\mp i \int_0^t \gamma(s;u)\,ds} u = e^{it\triangle_{\mathbb{S}^d}} f + v,$$

where by Theorem 1.3 and Corollary 1.4, we find that

$$e^{it\triangle_{\mathbb{S}^d}} f \in C^{p-\frac{d}{2}-}.$$

Moreover, $v \in \mathcal{Z}^{p-1/2+\varepsilon-}$ for some $\varepsilon \geq 1/2$ when $p > (d+1)/2$. It follows that $v \in C^{p-\frac{d}{2}-}$. Combining these two facts, we see that $u \in C^{p-\frac{d}{2}-}$, and hence

$$\dim_t(u) \leq (d+1) - \left(p - \tfrac{d}{2}\right).$$ □

Before proceeding to the proof of Theorem 1.5 we first derive a simple result that will guide our analysis. The restriction on the indices are due to the fact that $\kappa(n_1, n_2, n_3, n) = 0$ if any of the indices is strictly greater than the sum of the others. This restriction replaces the relation $n = n_1 - n_2 + n_3$ that one sees on the torus or real line.

Lemma 4.4. *Let $n_1, n_2, n_3, n \in \mathbb{N} \cup \{0\}$, $n_1 \geq n_3$,*

$$\max(n_1 - n_2 - n_3, n_2 - n_1 - n_3, 0) \leq n \leq n_1 + n_2 + n_3,$$

and define[4]

$$H(n_1,n_2,n_3,n) := n(n+d-1) - n_1(n_1+d-1) + n_2(n_2+d-1) - n_3(n_3+d-1).$$

Then at least one of the following must hold.

(1) $n = n_1$,
(2) $\langle n_1\rangle\langle n_2\rangle\langle n_3\rangle \gtrsim n^{3/2}$,
(3) $|H_n| \gtrsim \max(n_1,n_2)|n-n_1|$.

Proof. The proof is straightforward. We assume that all of the above are false. We first suppose that $n_2 \gg n_1$, so that the negation of the final item is immediately violated, as $H_n \gtrsim n_2^2$. We now assume $n_1 \gtrsim n_2$, which also implies $n_1 \gtrsim n$. Therefore the negation of item 2 implies that $n_2, n_3 \le \langle n_2\rangle\langle n_3\rangle \ll n^{1/2} \lesssim n_1^{1/2}$ as well as $n_1 \lesssim n$. Combining these facts we see that

$$|H_n| = |(n+n_1+d-1)(n-n_1) + n_2(n_2+d-1) - n_3(n_3+d-1)| \sim n_1|n-n_1|,$$

as the second two summands are, in magnitude, $\ll n$. This again contradicts the negation of the final item, completing the proof. □

The second case in Lemma 4.4 corresponds to resonances, which must be handled separately. In order to do that we need the following lemmas.

Lemma 4.5 (Szegö, [Sz, Theorem 8.21.13])**.** *Let* $0 < c < \pi$ *be fixed. Then (uniformly) for* $\theta \in [\frac{c}{n}, \pi - \frac{c}{n}]$, *we have*

$$P_n^{(\frac{d-2}{2},\frac{d-2}{2})}(\cos\theta) = n^{-\frac{1}{2}}k(\theta)\left(\cos(M\theta+\gamma) + \tfrac{O(1)}{n\sin\theta}\right),$$

where $M_n = n + \frac{d-1}{2}$, $\gamma = -\frac{d-1}{2}\cdot\frac{\pi}{2}$, *and*

$$k(\theta) = 2^{\frac{d-2}{2}}\pi^{-\frac{1}{2}}\sin(\theta)^{-\frac{d-1}{2}}.$$

In the remaining region we find that $|P_n^{(\frac{d-2}{2},\frac{d-2}{2})}(\cos\theta)| \sim n^{\frac{d-2}{2}}$.

Lemma 4.6. *Let* $0 < n_1, n_2 \le n$ *and* $d \ge 2$. *Then*

$$\frac{1}{\omega_d}\int_{\mathbb{S}^d} Y_{n_1}Y_{n_2}Y_n^2\,d\sigma = \frac{1}{\pi\omega_d}\int_0^\pi Y_{n_1}(\theta)Y_{n_2}(\theta)\,d\theta + O\Big(\tfrac{(n_1n_2)^{\frac{d-1}{2}+}}{n}\Big).$$

Moreover, for $n_1 \ge n_2 \ge n_3 \ge n_4$ *we have the estimate*

$$\frac{1}{\omega_d}\int_{\mathbb{S}^d} Y_{n_1}Y_{n_2}Y_{n_3}Y_{n_4}\,d\sigma = O\left((n_3n_4)^{\frac{d-2}{2}+}\right).$$

[4] $H(n_1,n_2,n_3,n)$ will often be abbreviated as H_n.

Remark 4.7. *Before moving on to the proof, we remark that the integrals on two sides are with respect to different measures; the one on the right hand side lacks the factor of* $\sin^{d-1}\theta$ *that arises due to the measure* $d\sigma$*.*

Proof. We begin first with a calculation in an attempt to understand the action of $(Y_n(\theta))^2\sin^{d-1}(\theta)$. Let

$$\alpha(n,d)=\frac{(2n+d-1)\Gamma(n+d-1)\Gamma(n+1)}{\Gamma(n+\frac{d}{2})^2}=(2n+d-1)\left(1+O\left(\tfrac{1}{n}\right)\right) \tag{28}$$

by Stirling's approximation (8). Then, for $\theta\in\left[\frac{1}{n},\pi-\frac{1}{n}\right]$, Lemma 4.5 and (28) give

$$\begin{aligned}Y_n(\theta)^2\sin^{d-1}(\theta)&=\frac{\alpha(n,d)}{n\pi}\left(\cos\left(\left(n+\tfrac{d-1}{2}\right)\theta-\tfrac{(d-1)\pi}{4}\right)+\tfrac{O(1)}{n\sin\theta}\right)^2\\&=\frac{2}{\pi}\cos\left(\left(n+\tfrac{d-1}{2}\right)\theta-\tfrac{(d-1)\pi}{4}\right)^2+O\left(\tfrac{1}{n\sin(\theta)}\right).\end{aligned}$$

A calculation for the cosine term above shows that it has a very strong localization near its mean:

$$\int_a^b\frac{2}{\pi}\cos\left(\left(n+\tfrac{d-1}{2}\right)\theta-\tfrac{(d-1)\pi}{4}\right)^2d\theta=\tfrac{1}{\pi}(b-a)+O\left(\tfrac{1}{d+n}\right). \tag{29}$$

We then let

$$\omega(\theta;n,d)=Y_n(\theta)^2\sin^{d-1}(\theta)-\tfrac{1}{\pi}, \tag{30}$$

and remark that for any $\frac{1}{n}<a<b<\pi-\frac{1}{n}$, ω satisfies

$$\int_a^b\omega(\theta;n,d)\,d\theta=O\left(\tfrac{1}{n^{1-}}\right), \tag{31}$$

where the implicit constants depend (harmlessly so) on the fixed d and the ε hidden in the $1-$ notation. We now note that, by symmetry, it suffices to consider

$$\int_0^{\frac{\pi}{2}}Y_{n_1}Y_{n_2}(Y_n)^2\sin(\theta)^{d-1}\,d\theta, \tag{32}$$

for $0<n_2\le n_1\le n$.

In what is to follow, our main tool will be Lemma 4.5. On $[0,\frac{1}{n}]$ we find that

$$Y_{n_1}Y_{n_2}(Y_n)^2=O\left((n_1n_2)^{\frac{d-1}{2}}n^{d-1}\right),$$

so that

$$\int_0^{\frac{1}{n}}Y_{n_1}Y_{n_2}(Y_n)^2\sin(\theta)^{d-1}\,d\theta\lesssim(n_1n_2)^{\frac{d-1}{2}}n^{d-1}\int_0^{\frac{1}{n}}\sin(\theta)^{d-1}\,d\theta\lesssim\frac{(n_1n_2)^{\frac{d-1}{2}}}{n}, \tag{33}$$

where the same inequality also holds for $\int_0^{1/n}Y_{n_1}Y_{n_2}d\theta$.

On $[\frac{1}{n},\frac{\pi}{2}]$, we must remove the mean. Specifically, we rewrite

$$\int_{\frac{1}{n}}^{\frac{\pi}{2}} Y_{n_1}Y_{n_2}(Y_n)^2\sin(\theta)^{d-1}\,d\theta = \int_{\frac{1}{n}}^{\frac{\pi}{2}} Y_{n_1}Y_{n_2}\omega(\theta;n,d)\,d\theta + \frac{1}{\pi}\int_{\frac{1}{n}}^{\frac{\pi}{2}} Y_{n_1}Y_{n_2}\,d\theta.$$

Therefore, it remains to prove that

$$\Big|\int_{\frac{1}{n}}^{\frac{\pi}{2}} Y_{n_1}Y_{n_2}\omega(\theta;n,d)\,d\theta\Big| \lesssim \frac{(n_1n_2)^{\frac{d-1}{2}}}{n}.$$

To utilize the average bound on $\omega(\theta;n,d)$, we need to apply integration by parts, for which we need the Jacobi polynomial identity [Sz, Equation 4.7.14]:

$$\partial_\theta P_n^{\frac{d-2}{2},\frac{d-2}{2}}(\cos(\theta)) = -\tfrac{n+d-1}{2}\sin(\theta)P_{n-1}^{\frac{d}{2},\frac{d}{2}}(\cos(\theta)), \tag{34}$$

from which, using Lemma 4.5, we find the following bounds for Y_n and $\partial_\theta Y_n$

$$|Y_n| \lesssim \frac{n^{\frac{d-1}{2}}}{\langle\theta n\rangle^{\frac{d-1}{2}}},\quad |\partial_\theta Y_n| \lesssim \frac{n^{\frac{d+3}{2}}\theta}{\langle\theta n\rangle^{\frac{d+1}{2}}}. \tag{35}$$

Applying integration by parts we see

$$\begin{aligned}\int_{\frac{1}{n}}^{\frac{\pi}{2}} Y_{n_1}Y_{n_2}\omega(\theta;n,d)\,d\theta &= Y_{n_1}(\pi/2)Y_{n_2}(\pi/2)\int_{\frac{1}{n}}^{\frac{\pi}{2}}\omega(s;n,d)\,ds \\ &\quad - \int_{\frac{1}{n}}^{\frac{\pi}{2}}\partial_\theta\left(Y_{n_1}Y_{n_2}\right)\int_{\frac{1}{n}}^{\theta}\omega(s;n,d)\,ds\,d\theta.\end{aligned} \tag{36}$$

By (31), we bound this by

$$\lesssim \frac{1}{n^{1-}} + \frac{n_1^{\frac{d-1}{2}}n_2^{\frac{d-1}{2}}}{n^{1-}}\int_{\frac{1}{n}}^{\frac{\pi}{2}}\frac{n_1}{\langle\theta n_1\rangle^{\frac{d-1}{2}}\langle\theta n_2\rangle^{\frac{d-1}{2}}}d\theta \lesssim \frac{(n_1n_2)^{\frac{d-1}{2}}}{n^{1-}}.$$

To obtain the last inequality, consider the integrals on $(\frac{1}{n},\frac{1}{n_1})$, $(\frac{1}{n_1},\frac{1}{n_2})$, $(\frac{1}{n_2},\frac{\pi}{2})$ separately. □

Writing the nonlinearity, $|u|^2u$, on the Fourier side, $u=\sum_{n=0}^\infty \widehat{u}_nY_n$, we have

$$\mathcal{F}(|u|^2u)(n) = \sum_{n_1,n_2,n_3}\widehat{u}_{n_1}\overline{\widehat{u}_{n_2}}\widehat{u}_{n_3}\kappa(n_1,n_2,n_3,n),$$

where κ is as in (26). We split the resonant portion, $n_1=n$ or $n_3=n$, into

$$2\widehat{u}_n\sum_{n_2,n_3}\overline{\widehat{u}_{n_2}}\widehat{u}_{n_3}\kappa(n,n,n_2,n_3) - \widehat{u}_n^2\sum_{n_2}\overline{\widehat{u}}_{n_2}\kappa(n,n,n_2,n),$$

where the second term is not only, of course, easy to handle down to the local well-posedness level, but also presents another large frequency to aid in smoothing. The first, however, is of the form

$$\widehat{u}_n\frac{2}{\pi\omega_d}\sum_{n_2,n_3}\overline{\widehat{u}_{n_2}}\widehat{u}_{n_3}\int_0^\pi Y_{n_2}(\theta)Y_{n_3}(\theta)\,d\theta + \text{smoother},$$

by Lemma 4.6. The first of these terms is analagous to the $\|u\|_{L^2_x}$ term that appears in the smoothing statement of [ErTz2], in that it is $\widehat{u}_n$ multiplied by a *real* function of time and the solution. Motivated by this, we define the change of variables given by

$$\widehat{u}_n = e^{\pm i\gamma(t;v)}\widehat{v}_n, \quad \text{where} \tag{37}$$

$$\gamma(t;v) = \frac{2}{\pi\omega_d}\sum_{k,\ell}\overline{\widehat{v}_k}(t)\widehat{v}_\ell(t)\int_0^\pi Y_k(\theta)Y_\ell(\theta)\,d\theta, \tag{38}$$

which acts as a phase rotation dependent only on time and the solution. Moreover, because of the conjugate in the definition we see that this is easily invertible. The resulting equation is then given by

$$\begin{cases} i\partial_t v + \triangle_{\mathbb{S}^d} v \pm v(|v|^2 - \gamma(t;v)) = 0 \\ v(x,0) = u_0(x). \end{cases} \tag{39}$$

Before we consider the Fourier transform of the new nonlinearity, $N(v) := \pm v(|v|^2 - \gamma(t;v))$, we consider the sets given in Lemma 4.4 and define:

$$\Lambda_0(n) = \{(n_1,n_2,n_3)\in\mathbb{N}_0^3 : n_1 = n \text{ or } n_3 = n\}$$
$$\Lambda_1(n) = \{(n_1,n_2,n_3)\in\mathbb{N}_0^3\setminus\Lambda_0(n) : \langle n_1\rangle\langle n_2\rangle\langle n_3\rangle \gtrsim n^{3/2}\}$$
$$\Lambda_2(n) = \{(n_1,n_2,n_3)\in\mathbb{N}_0^3\setminus(\Lambda_0(n)\cup\Lambda_1(n)) : |H_n| \gtrsim \max(n_1,n_2,n_3)|n-\max(n_1,n_3)|\}.$$

We note that these sets are disjoint for all $n\in\mathbb{N}_0$ and that these sets directly correspond to frequency configurations highlighted in Lemma 4.4. In particular, the set $\Lambda_2(n)$ contains the indices when the phase, H_n, is large. In order to use the fact that we have large modulation, we need to apply differentiation by parts for the contribution of these terms. Before proceeding, we note that we've truncated the summation notation, opting to drop the $(n_1,n_2,n_3)\in\Lambda_i(n)$ in favor of simply stating $\Lambda_i(n)$, as there is no confusion.

The Fourier coefficients of the nonlinearity, $N(v) := \pm v(|v|^2 - \gamma(t;v))$, are given by

$$\begin{aligned}\widehat{N(v)}(n) &= \pm 2\widehat{v}_n\sum_{n_2,n_3}\overline{\widehat{v}_{n_2}}\widehat{v}_{n_3}\left(\kappa(n,n,n_2,n_3) - \frac{1}{\pi\omega_d}\int_0^\pi Y_{n_2}(\theta)Y_{n_3}(\theta)\,d\theta\right)\\ &\mp \widehat{v}_n^2\sum_{n_2}\overline{\widehat{v}}_{n_2}\kappa(n,n,n_2,n) \pm \sum_{\Lambda_1(n)}\widehat{v}_{n_1}\overline{\widehat{v}_{n_2}}\widehat{v}_{n_3}\kappa(n,n_1,n_2,n_3) \pm \sum_{\Lambda_2(n)}\widehat{v}_{n_1}\overline{\widehat{v}_{n_2}}\widehat{v}_{n_3}\kappa(n,n_1,n_2,n_3)\\ &=: \widehat{N_{0,1}(v)}(n) + \widehat{N_{0,2}(v)}(n) + \widehat{N_1(v)}(n) + \widehat{N_2(v)}(n).\end{aligned}$$

By differentiation by parts, applied as in [EGT, Proposition 6.1] only to N_2, the solution of (39) satisfies

$$i\partial_t\left(e^{-it\Delta}v - e^{-it\Delta}B(v)\right) = -e^{-it\Delta}\big(N_{0,1}(v) + N_{0,2}(v) + N_1(v) + N_{2,1}(v) + N_{2,2}(v)\big), \tag{40}$$

where

$$\widehat{B(v)}(n) = \pm i \sum_{\Lambda_2(n)} \frac{1}{H_n} \widehat{v}_{n_1} \overline{\widehat{v}_{n_2}} \widehat{v}_{n_3} \kappa(n, n_1, n_2, n_3),$$

$$\widehat{N_{2,1}(v)}(n) = \pm 2i \sum_{\Lambda_2(n)} \frac{1}{H_n} \widehat{w}_{n_1} \overline{\widehat{v}_{n_2}} \widehat{v}_{n_3} \kappa(n, n_1, n_2, n_3),$$

$$\widehat{N_{2,2}(v)}(n) = \mp i \sum_{\Lambda_2(n)} \frac{1}{H_n} \widehat{v}_{n_1} \overline{\widehat{w}_{n_2}} \widehat{v}_{n_3} \kappa(n, n_1, n_2, n_3).$$

Here

$$w = ie^{it\Delta}[\partial_t(e^{-it\Delta}v)] = -N(v) = \mp v(|v|^2 - \gamma(t; v)).$$

A priori estimates for the terms $B(v)$, $N_1(v)$, $N_{2,1}(v)$, and $N_{2,2}(v)$ will be given in Lemma 4.11 below. The term $N_{0,1}(v)$ will be estimated in Lemma 4.12, and the term $N_{0,2}(v)$ in Lemma 4.13.

The following proposition is a repeatedly used application of Young's inequality (or just Cauchy-Schwarz), which is best to state once and use by reference.

Proposition 4.8. *Let $0 \le \delta \le 1$. Then for any $\{a_n\}$, $\{b_m\}$ we have for any $\eta > \frac{1-\delta}{2}$,*

$$\Big| \sum_{n,m \ge 0} \frac{a_n b_m}{\langle n - m \rangle^\delta} \Big| \lesssim \|a_n \langle n \rangle^\eta\|_{\ell^2_n} \|b_m \langle m \rangle^\eta\|_{\ell^2_m}.$$

Before proceeding, we define the space within which the wellposedness arguments are done– the standard Bourgain space adapted to $\mathbb{S}^d$:

$$\|u\|_{X^{s,b}} := \|\langle n \rangle^s \langle \tau + n(n+d-1) \rangle^b |\mathcal{F}_{x,t} u(n,\tau)| \|_{L^2_\tau \ell^2_n},$$

$$\|u\|_{X^{s,b}_T} := \inf_{w|_{[0,T]} = u|_{[0,T]}} \|w\|_{X^{s,b}}.$$

An easy consequence of the definition of these spaces and Proposition (4.8) is that the phase rotation (37) is well defined. That is, $\int_0^t \gamma(s, v)\, ds$ is finite for $v \in X^{\frac{d-2}{2}+, 1/2+}$:

Proposition 4.9. *For $0 < t < T$, we have*

$$(41) \qquad \Big| \int_0^t \sum_{k,\ell} \overline{\widehat{v}_k}(s) \widehat{v}_\ell(s) \int_0^\pi Y_k(\theta) Y_\ell(\theta)\, d\theta\, ds \Big| \lesssim \|v\|^2_{X^{\frac{d-2}{2}+, 1/2+}},$$

where the implicit constant depends on T only.

Proof. We first note that by (35) we have $\int_0^\pi Y_k(\theta) Y_\ell(\theta)\, d\theta = O((k\ell)^{\frac{d-2}{2}+})$. Therefore the contribution of the terms $k = \ell$ is bounded by $T\|v\|^2_{C^0_{t\in[0,T]} H_x^{\frac{d-2}{2}+}}$, which suffices. For the terms $k \ne \ell$, by Plancherel, and noting that $|\widehat{\chi_{[0,t]}}(\tau)| \lesssim \frac{1}{\langle \tau \rangle}$ with an implicit constant depending on T only, we have the bound

$$\sum_{k\neq\ell}\int_{\mathbb{R}^2}\frac{|\mathcal{F}_{x,t}v(\ell,\tau_1)||\mathcal{F}_{x,t}v(k,\tau)|(k\ell)^{\frac{d-2}{2}+}}{\langle\tau-\tau_1\rangle}d\tau_1 d\tau$$
$$\lesssim \|v\|^2_{X^{\frac{d-2}{2}+,1/2+}}\Big[\sum_{k\neq\ell}\int_{\mathbb{R}^2}\frac{d\tau_1 d\tau}{\langle\tau-\tau_1\rangle^2\langle\tau-k(k+d-1)\rangle^{1+}\langle\tau_1-\ell(\ell+d-1)\rangle^{1+}}\Big]^{1/2}$$
$$\lesssim \|v\|^2_{X^{\frac{d-2}{2}+,1/2+}}\Big[\sum_{k\neq\ell}\frac{1}{\langle k(k+d-1)-\ell(\ell+d-1)\rangle^{1+}}\Big]^{1/2}\lesssim \|v\|^2_{X^{\frac{d-2}{2}+,1/2+}}.$$

In the second inequality we used Cauchy-Schwarz in all variables and the definition of $X^{s,b}$ norm. □

We also need another proposition which is a bilinear Strichartz estimate that follows from (5.1) and the bilinear form of (16), see [BGZ2, Proposition 4.3].

Proposition 4.10. *For $N\geq M$ dyadic and all $\varepsilon>0$, we have*

$$\|P_N(\eta)P_M(\nu)\|_{L^2_{t\in[0,2\pi]}L^2_x}\lesssim M^{\frac{d-2}{2}+\varepsilon}\|P_N(\eta)\|_{X^{0,1/2-}}\|P_M(\nu)\|_{X^{0,1/2-}},$$

and hence for any $\alpha,\beta\geq 0$ satisfying $\alpha+\beta>\frac{d-2}{2}$, we have

$$\|\eta\nu\|_{L^2_{t\in[0,2\pi]}L^2_x}\lesssim\|\eta\|_{X^{\alpha,1/2-}}\|\nu\|_{X^{\beta,1/2-}}$$

The rest of this section consists of the required estimates for the terms appearing in (40).

Lemma 4.11. *Let $d\geq 2$, $s>\frac{d-2}{2}$, and $0\leq\varepsilon<\frac{1}{2}\min\big(s-\frac{d-2}{2},2\big)$, then*

$$\|N_{2,1}(v)\|_{X_T^{s+\varepsilon,-1/2+}}+\|N_{2,2}(v)\|_{X_T^{s+\varepsilon,-1/2+}}\lesssim_\varepsilon\|v\|^5_{X_T^{s,1/2+}},\tag{42}$$

$$\|N_1(v)\|_{X_T^{s+\varepsilon,-1/2+}}\lesssim_\varepsilon\|v\|^3_{X_T^{s,1/2+}},\tag{43}$$

$$\|B(v)\|_{C^0_tH^{s+\varepsilon}_x}\lesssim_\varepsilon\|v\|^3_{C^0_tH^s_x}.\tag{44}$$

Proof. We first handle the term $N_{2,1}(v)$, neglecting the term $N_{2,2}(v)$ as it is proved similarly. We then split into when we have $v|v|^2$ and $v\gamma(t;v)$, separately.

When we have $v|v|^2$, using the restriction imposed by $\Lambda_2(n)$, we have

$$|H_n|\gtrsim\max(n_1,n_2,n_3)|n-\max(n_1,n_3)|.$$

Similarly, we find by the disjointness of $\Lambda_i(n)$ for $i\in\{1,2\}$ and the fact that at least one $n_j\gtrsim n$ for $j\in\{1,2,3\}$ that $\langle n_{i_1}\rangle\langle n_{i_2}\rangle\ll n^{1/2}$ for some $\{i_1,i_2\}\subset\{1,2,3\}$ and $i_1\neq i_2$. We now ignore the presence of conjugates[5] and consider two cases:

$$\text{i) }\{n_2,n_3\}=\{i_1,i_2\}\qquad\text{ii) }\{n_1,n_3\}=\{i_1,i_2\}.$$

[5] This doesn't create any issues since in Proposition 4.10 we can replace η and/or ν with their conjugates on the left hand side of the inequalities.

In the first case we find that $n_1 \sim n$, and hence $|H_n| \gg n_1 \gtrsim \langle n_2\rangle\langle n_3\rangle$. Relabeling

$$\mathcal{F}_{x,t}(\varphi)(n,\tau) = |\mathcal{F}_{x,t}(v)(n,\tau)| \qquad \text{and} \qquad \mathcal{F}_{x,t}(\nu)(n,\tau) = \mathcal{F}_{x,t}\left(\varphi^3\right)(n,\tau)$$

and noting that

$$\frac{\langle n\rangle^{\varepsilon}\langle n_2\rangle^{\frac{d}{2}+}\langle n_3\rangle^{\frac{d}{2}+}}{|H_n|\langle n_2\rangle^s\langle n_3\rangle^s} \lesssim \langle n_2\rangle^{\frac{d}{2}-s-1+\varepsilon+}\langle n_3\rangle^{\frac{d}{2}-s-1+\varepsilon+} \lesssim 1$$

for $\varepsilon < \max\left(s-\frac{d-2}{2},1\right)$, we see that it is sufficient bound

$$\Big\|\mathcal{F}_n^{-1}\Big(\sum_{\Lambda_2(n)}\langle n_1\rangle^s\widehat{\nu}_{n_1}\langle n_2\rangle^{s-\frac{d}{2}+}\widehat{\varphi}_{n_2}\langle n_3\rangle^{s-\frac{d}{2}+}\widehat{\varphi}_{n_3}\kappa(n,n_1,n_2,n_3)\Big)\Big\|_{X_T^{0,-1/2+}}.$$

By using the positivity of the space-time Fourier transforms, we may expand the summation from $\Lambda_2(n)$ to $\mathbb{N}_0^3$, use

$$\langle n_1\rangle^s\widehat{\nu}_{n_1} \lesssim [\widehat{\varphi^2 J_x^s\varphi}]_{n_1},$$

invoke duality for $w \in X^{0,1/2-}$, and apply Parseval's to find

$$\begin{aligned}
&\Big\|\mathcal{F}_n^{-1}\Big(\sum_{(n_1,n_2,n_3)\in\mathbb{N}_0}\langle n_1\rangle^s\widehat{\nu}_{n_1}\langle n_2\rangle^{s-\frac{d}{2}+}\widehat{\varphi}_{n_2}\langle n_3\rangle^{s-\frac{d}{2}+}\widehat{\varphi}_{n_3}\kappa(n,n_1,n_2,n_3)\Big)\Big\|_{X_T^{0,-1/2+}} \\
&\qquad \lesssim \int_{\mathbb{R}}\int_{\mathbb{S}^d}\left|w\varphi^2 J_x^s\varphi\big(J_x^{s-\frac{d}{2}+}\varphi\big)^2\right| dt d\sigma(x) \lesssim \|w\varphi^2 J_x^s\varphi\|_{L^1_{x,t}}\|J^{s-\frac{d}{2}+}\varphi\|^2_{L^\infty_{t,x}} \\
&\qquad\qquad \lesssim \|w\varphi\|_{L^2_{x,t}}\|\varphi J_x^s\varphi\|_{L^2_{x,t}}\|\varphi\|_{X_T^{s,1/2+}} \lesssim \|v\|^5_{X_T^{s,1/2+}},
\end{aligned}$$

by the bilinear L^2 estimate (Proposition 4.10) and Sobolev embedding.

In the second case, $\{n_1,n_3\} = \{i_1,i_2\}$, we have $n_2 \sim n$ and $n_2^{1/2} \gtrsim \langle n_1\rangle\langle n_3\rangle$. In particular, $|H_n| \gtrsim n_2$ (in fact we have $|H_n| \gtrsim n_2^2$, but we only use n_2 in order for the argument to handle the case that $n_3 \sim n$). Therefore

$$\frac{\langle n\rangle^{\varepsilon}}{|H_n|} \lesssim \frac{1}{n_2^{1-\varepsilon}} \lesssim \frac{1}{\langle n_1\rangle^{2-2\varepsilon}\langle n_3\rangle^{2-2\varepsilon}},$$

we expand the summation from $\Lambda_2(n)$ to $\mathbb{N}_0^3$, and find:

$$\lesssim \Big\|\mathcal{F}_n^{-1}\Big(\sum_{(n_1,n_2,n_3)\in\mathbb{N}_0}\langle n_1\rangle^{2\varepsilon-2}\widehat{\nu}_{n_1}\langle n_2\rangle^s\widehat{\varphi}_{n_2}\langle n_3\rangle^{2\varepsilon-2}\widehat{\varphi}_{n_3}\kappa(n,n_1,n_2,n_3)\Big)\Big\|_{X_T^{0,-1/2+}} \tag{45}$$

Note that

$$\widehat{\nu}_n = \sum_{n_1,n_2,n_3}\widehat{\varphi}_{n_1}\widehat{\varphi}_{n_2}\widehat{\varphi}_{n_3}\kappa(n,n_1,n_2,n_3).$$

By the support condition of κ and symmetry, it suffices to consider the cases $n \ll n_1 \approx n_2 \gtrsim n_3$ and $n \approx n_1 \gtrsim n_2, n_3$. Denoting the contributions of these terms as ν^h, ν^ℓ, respectively, we see that

$$\mathcal{F}(J_x^\alpha\nu^h)(n) \lesssim \mathcal{F}[(J_x^{\alpha/2}\varphi)^2\varphi](n), \quad \text{for } \alpha \geq 0, \text{ and} \tag{46}$$

$$\mathcal{F}(J_x^\alpha \nu^\ell)(n) \lesssim \mathcal{F}[(J_x^\alpha \varphi)\varphi^2](n), \quad \text{for all } \alpha \in \mathbb{R}. \tag{47}$$

Using these we estimate the contributions of ν^h and ν^ℓ to (45) as follows. For ν^h, using (46) with $\alpha = d + 2\epsilon - 2+$, and duality, it suffices to bound

$$\int_{\mathbb{R}} \int_{\mathbb{S}^d} \left| w J_x^{-d-} [\varphi (J_x^{\varepsilon + \frac{d-2}{2}+} \varphi)^2] J_x^s \varphi J_x^{2\varepsilon - 2} \varphi \right| dt d\sigma(x)$$
$$\leq \|w J_x^{2\varepsilon-2}\varphi\|_{L^2_{x,t}} \left\| J_x^{-d-} [\varphi (J_x^{\varepsilon + \frac{d-2}{2}+} \varphi)^2] \right\|_{L^2_t L^\infty_x} \|J_x^s \varphi\|_{L^\infty_t L^2_x}$$
$$\lesssim \|w\|_{X^{0,1/2-}} \|\varphi\|^2_{X^{s,1/2+}} \left\| \varphi (J_x^{\varepsilon + \frac{d-2}{2}} \varphi)^2 \right\|_{L^2_t L^1_x}.$$

In the last step we used the bilinear L^2 estimate and sobolev embedding. The following inequality finishes the proof for the contribution of ν^h:

$$\left\| \varphi (J_x^{\varepsilon + \frac{d-2}{2}+} \varphi)^2 \right\|_{L^2_t L^1_x} \leq \|\varphi J_x^{\varepsilon + \frac{d-2}{2}+} \varphi\|_{L^2_{t,x}} \|J_x^{\varepsilon + \frac{d-2}{2}+} \varphi\|_{L^\infty_t L^2_x} \lesssim \|\varphi\|^3_{X^{s,1/2+}}.$$

Above we used the bilinear L^2 estimate and that $\epsilon < s - \frac{d-2}{2}$.

For the contribution of ν^ℓ to (45), using (46) with $\alpha = 2\epsilon - 2$, duality, bilinear L^2 estimate, and Sobolev embedding, we have the required bound:

$$\int_{\mathbb{R}} \int_{\mathbb{S}^d} \left| w \varphi^2 J_x^{2\varepsilon-2} \varphi J_x^s \varphi J_x^{2\varepsilon-2} \varphi \right| dt d\sigma(x)$$
$$\leq \|w\varphi\|_{L^2_{x,t}} \|\varphi J_x^s \varphi\|_{L^2_{x,t}} \|J_x^{2\varepsilon-2}\varphi\|^2_{L^\infty_{x,t}} \lesssim \|w\|_{X^{0,1/2-}} \|\varphi\|^5_{X^{s,1/2+}}.$$

We now consider the contribution of $v\gamma(t;v)$ to $N_{2,1}(v)$. Recall that

$$\gamma(t;v) = \frac{2}{\pi \omega_d} \sum_{k,\ell} \overline{\widehat{v}_k}(t) \widehat{v}_\ell(t) \int_0^\pi Y_k(\theta) Y_\ell(\theta)\, d\theta.$$

We first need to understand the Fourier transform of γ in t. Defining φ as above and noting that (35) yields $\int_0^\pi Y_k Y_\ell \, d\theta = O((k\ell)^{\frac{d-2}{2}+})$, we have (for each n)

$$\left| \mathcal{F}_t(\gamma(t;v))(\tau) \right| \lesssim \int \sum_{k \neq \ell < n} \mathcal{F}_{x,t}(\bar{\varphi})(k, \tau_1) \mathcal{F}_{x,t}(\varphi)(\ell, \tau - \tau_1)(k\ell)^{\frac{d-2}{2}+} d\tau_1$$
$$+ \int \sum_{\substack{\max(k,\ell) \geq n \\ k \neq \ell}} \mathcal{F}_{x,t}(\bar{\varphi})(k, \tau_1) \mathcal{F}_{x,t}(\varphi)(\ell, \tau - \tau_1)(k\ell)^{\frac{d-2}{2}+} d\tau_1 + \mathcal{F}_t(\|\varphi\|^2_{H_x^{\frac{d-2}{2}+}})$$
$$= \mathcal{F}_t \Big(\sum_{k \neq \ell < n} \widehat{\bar{\varphi}}_k \widehat{\varphi}_\ell (k\ell)^{\frac{d-2}{2}+} + \sum_{\substack{\max(k,\ell) \geq n \\ k \neq \ell}} \widehat{\bar{\varphi}}_k \widehat{\varphi}_\ell (k\ell)^{\frac{d-2}{2}+} + \|\varphi\|^2_{H_x^{\frac{d-2}{2}+}} \Big)(\tau)$$
$$=: \mathcal{F}_t \big(\gamma_{1,n} + \gamma_{2,n} + \|\varphi\|^2_{H_x^{\frac{d-2}{2}+}} \big)(\tau) =: \mathcal{F}_t(\Gamma_n)(\tau),$$

where $\mathcal{F}_{x,t}(\bar{\varphi})(k,\tau_1) = \mathcal{F}_{x,t}(\varphi)(k, -\tau_1) \geq 0$. With this bound the contribution of $v\gamma(t;v)$ boils down to estimating

$$(48)\quad \Big\|\mathcal{F}_n^{-1}\Big(\sum_{\Lambda_2(n)}\frac{\langle n\rangle^{s+\varepsilon}\Gamma_n}{|H_n|}\widehat{\varphi}_{n_1}\widehat{\bar{\varphi}}_{n_2}\widehat{\varphi}_{n_3}\kappa(n,n_1,n_2,n_3)\Big)\Big\|_{X^{0,-1/2+}} =$$

$$\Big\|\int\sum_{\Lambda_2(n)}\frac{\langle n\rangle^{s+\varepsilon}\mathcal{F}_t(\Gamma_n)(\tau_0)\mathcal{F}_{x,t}\varphi(n_1,\tau_1)\mathcal{F}_{x,t}\bar{\varphi}(n_2,\tau_2)\mathcal{F}_{x,t}\varphi(n_3,\tau_3)}{\langle\tau+n(n+d-1)\rangle^{\frac12-}|H_n|}\kappa(n,n_1,n_2,n_3)d\tau_1d\tau_2d\tau_3\Big\|_{L^2_\tau\ell^2_n},$$

where $\tau_0=\tau-\tau_1-\tau_2-\tau_3$.

For the contribution of $\|\varphi\|^2_{H_x^{\frac{d-2}{2}+}}$ to (48), noting that $n\lesssim\max(n_1,n_2,n_3)$, $\frac{n^\varepsilon}{|H_n|}\lesssim 1$, and using duality, it suffices to observe that

$$\big\|w\varphi^2J^s\varphi\|\varphi\|^2_{H_x^{\frac{d-2}{2}+}}\big\|_{L^1_{x,t}}\lesssim\|w\varphi\|_{L^2_{x,t}}\|\varphi J^s\varphi\|_{L^2_{x,t}}\|\varphi\|^2_{C_t^0H_x^{\frac{d-2}{2}+}}\lesssim\|w\|_{X^{0,1/2-}}\|\varphi\|^5_{X^{s,\frac12+}}.$$

For the contribution of $\gamma_{1,n}$, first note that (since $k,\ell\le n$ and $|H_n|\gtrsim n$)

$$\frac{n^\varepsilon(k\ell)^{\frac{d-2}{2}+}}{|H_n|}\lesssim(k\ell)^{s-\frac12-},$$

for $\varepsilon<\frac12\min\big(s-\frac{d-2}{2},2\big)$. Therefore, using Proposition 4.8 in the k,ℓ sums with $\delta=0$, we see that

$$\frac{\langle n\rangle^\varepsilon\Gamma_n}{|H_n|}\lesssim\|\varphi\|^2_{H^s_x}.$$

The bound then is identical to the contribution of $\|\varphi\|^2_{H_x^{\frac{d-2}{2}+}}$ above.

To handle the contribution of the $\gamma_{2,n}$ term, we consider two subcases:

$$|H_n|\gtrsim(k+\ell+d-1)|k-\ell|\quad\text{and}\quad|H_n|\ll(k+\ell+d-1)|k-\ell|.$$

In the first case, we recall that $\max(k,\ell)\gtrsim n$, $k\neq\ell$, and hence

$$(49)\qquad \frac{\langle n\rangle^\varepsilon(k\ell)^{\frac{d-2}{2}}}{|H_n|}\lesssim\frac{(k\ell)^{\frac{d-2}{2}}}{|k-\ell|}$$

for $\varepsilon<1$, so that by Proposition 4.8 with $\delta=1$ we find

$$(50)\qquad \frac{\langle n\rangle^\varepsilon\Gamma_n}{|H_n|}\lesssim\|\varphi\|^2_{H^s_x},$$

for $s>\frac{d-2}{2}$. We see that the bound is again identical to the prior contributions.

When $|H_n|\ll(k+\ell+d-1)|k-\ell|$, we note that

$$|H_n+k(k+d-1)-\ell(\ell+d-1)|\gtrsim(k+\ell)|k-\ell|,$$

and hence by considering the weights in the $X^{s,b}$ norms in (48) we see

$$\sigma:=\langle\tau+n(n+d-1)\rangle+\sum_{i=1}^3\langle\tau_i+n_i(n_i+d-1)\rangle+\langle\lambda_1+k(k+d-1)\rangle$$
$$+\langle\lambda_2+\ell(\ell+d-1)\rangle\gtrsim(k+\ell)|k-\ell|,$$

so that one of the summands on the left hand side of the above must be $\gtrsim (k+\ell)|k-\ell|$. From this, we split into four subcases:

i) $\langle \tau + n(n+d-1)\rangle \gtrsim (k+\ell)|k-\ell|$,

ii) $\langle \tau_i + (-1)^i n_i(n_i+d-1)\rangle \gtrsim (k+\ell)|k-\ell|$ for $1 \le i \le 3$,

iv) $\langle \lambda_1 - k(k+d-1)\rangle \gtrsim (k+\ell)|k-\ell|$,

and let $\sigma_0 = \langle \tau + n(n+d-1)\rangle$ and $\sigma_i = \langle \tau_i + (-1)^i n_1(n_1+d-1)\rangle$ for $1 \le i \le 3$. We also note that for $\varepsilon < \frac{1}{2}(s - \frac{d-2}{2})$ we have

$$\langle n\rangle^{\varepsilon}(k\ell)^{\frac{d-2}{2}+} \lesssim \sigma^{1/2-}\frac{(k\ell)^{s-\frac{1}{4}-}}{|k-\ell|^{1/2-}}. \tag{51}$$

To handle the first two cases we then have by Proposition 4.8 with $\delta = 1/2-$ that

$$\langle n\rangle^{\varepsilon}\Gamma_n \lesssim \big(\sum_{i=0}^{3}\sigma_i\big)^{1/2-}\|\mathcal{F}_t\varphi\|^2_{H^s_x},$$

and hence by Young's and the $\Lambda_2(n)$ restriction it is sufficient to bound

(52)
$$\Big\|\int\sum_{\Lambda_2(n)}\frac{\langle n_1\rangle^s\sigma_i^{1/2-}\mathcal{F}_{x,t}\varphi(n_1,\tau_1)\mathcal{F}_{x,t}\bar{\varphi}(n_2,\tau_2)\mathcal{F}_{x,t}\varphi(n_3,\tau_3)}{\langle \tau + n(n+d-1)\rangle^{\frac{1}{2}-}\langle n_2\rangle\langle n_3\rangle}\kappa(n,n_1,n_2,n_3)d\tau_1 d\tau_2 d\tau_3\Big\|_{L^2_\tau \ell^2_n},$$

for $0 \le i \le 3$. Note that we have harmlessly assumed that $n_1 \gtrsim n_2, n_3$ in the above display.

We now use Plancherel, duality with $w \in X^{0,1/2-}$, and Hölders to find

$$\begin{aligned}(52) \lesssim{}& \|\sigma_0 w\|_{L^2_{x,t}}\|J^s_x\varphi\|_{L^2_{x,t}}\|J^{-1}_x\varphi\|_{L^\infty_{x,t}} + \|w\|_{L^2_{x,t}}\|J^s_x\sigma_1\varphi\|_{L^2_{x,t}}\|J^{-1}_x\varphi\|_{L^\infty_{x,t}}\\ &+ \|w\|_{L^2_{x,t}}\|J^s_x\varphi\|_{L^2_{x,t}}\|J^{-1}_x\sigma_2\varphi\|_{L^2_t L^\infty_x}\|J^{-1}_x\varphi\|_{L^\infty_{x,t}} \lesssim \|\varphi\|^3_{X^{s,1/2+}},\end{aligned}$$

for $s > \frac{d-2}{2}$.

To handle the last case, we note again that by Youngs, (51), and proposition 4.8 with $\delta = 1/2-$, that we have

$$\begin{aligned}&\Big\|\langle n\rangle^{\varepsilon}\sum_{\substack{\max(k,\ell)\ge n\\ k\ne\ell}}\int \mathcal{F}_{x,t}\bar{\varphi}(k,\lambda_1)\mathcal{F}_{x,t}\varphi(\ell,\tau-\lambda_1)\,d\lambda_1\Big|\int_0^\pi Y_k(\theta)Y_\ell(\theta)\,d\theta\Big|\Big\|_{L^{2-}_\tau} \\ &\lesssim \sum_{\substack{\max(k,\ell)\ge n\\ k\ne\ell}}\|\langle \tau - k(k+d-1)\rangle^{1/2-}\mathcal{F}_{x,t}\bar{\varphi}(k,\tau)\|_{L^{2-}_\tau}\|\mathcal{F}_{x,t}\varphi(\ell,\tau)\|_{L^1_\tau}\frac{(k\ell)^{s-1/4-}}{|k-\ell|^{1/2-}} \\ &\lesssim \|\langle k\rangle^s\langle \tau - k(k+d-1)\rangle^{1/2+}\mathcal{F}_{x,t}\bar{\varphi}(k,\tau)\|_{L^2_\tau\ell^2_k}\|\langle \ell\rangle^s\mathcal{F}_{x,t}\varphi(\ell,\lambda_2)\|_{\ell^2_\ell L^1_\tau},\end{aligned} \tag{53}$$

for $s > \frac{d-2}{2}$ and $\varepsilon < 1/2(s - \frac{d-2}{2})$. In other words, the contribution to Γ_n can be estimated by

$$\|\langle n\rangle^{\varepsilon}\Gamma_n(\tau)\|_{\ell^\infty_n L^{2-}_\tau} \lesssim \|\varphi\|^2_{X^{s,1/2+}}.$$

We now estimate (48) by assuming that $n_1 \sim n$ and using $H_n \gtrsim \langle n_2\rangle\langle n_3\rangle$ to write

$$(48) \lesssim \| \frac{1}{\langle \tau + n(n+d-1)\rangle^{1/2-}} (f *_\tau \mathcal{F}_t \Gamma_n) \|_{L^2_\tau \ell^2_n} \lesssim \| f *_\tau \mathcal{F}_t \Gamma_n \|_{\ell^2_n L^{\infty-}_\tau}$$
$$\lesssim \big\| \|f\|_{L^2_\tau} \|\Gamma_n\|_{L^{2-}_\tau} \big\|_{\ell^2_n} \lesssim \|f\|_{L^2_n L^2_\tau} \|\Gamma_n\|_{\ell^\infty_n L^{2-}_\tau} \lesssim \|f\|_{L^2_n L^2_\tau} \|\varphi\|^2_{X^{s,1/2+}},$$

where

$$f = \int \sum_{\Lambda_2(n)} \langle n_1\rangle^s \mathcal{F}_{x,t}\varphi(n_1,\tau_1) \frac{\mathcal{F}_{x,t}\bar{\varphi}(n_2,\tau_2)}{\langle n_2\rangle} \frac{\mathcal{F}_{x,t}\varphi(n_3,\tau_3)}{\langle n_3\rangle} \kappa(n,n_1,n_2,n_3) d\tau_1 d\tau_2 d\tau_3.$$

Owing to Plancherel and Sobolev embedding we find that

$$\|f\|_{L^2_\tau \ell^2_n} \lesssim \|\varphi\|_{X^{s,1/2+}} \|J_x^{-1}\varphi\|^2_{X^{\frac{d}{2}+,1/2+}} = \|\varphi\|_{X^{s,1/2+}} \|\varphi\|_{X^{\frac{d-1}{2}+,1/2+}},$$

and the full bound follows.

The bound (43) follows from Lemma 4.4 and the summation restriction on $\Lambda_1(n)$, as we'll have $\langle n_1\rangle\langle n_2\rangle\langle n_3\rangle \gtrsim n^{3/2}$. Specifically, we ignore the presence of conjugates because we will apply the bilinear L^2 estimate and and assume that either $n_1 \sim n$ or $n_1 \sim n_2 \gg n$. In either case we must have that $\langle n_2\rangle\langle n_3\rangle \gtrsim n^{1/2}$.

It then follows by (47), duality with $w \in X^{0,1/2-}$, and the bilinear L^2 estimate that

$$\|J^s\varphi(J^{2\varepsilon}\varphi)\|_{X^{0,-1/2+}} \lesssim \|wJ^{2\varepsilon}\varphi\|_{L^2_{x,t}} \|J^s\varphi J^{2\varepsilon}\varphi\|_{L^2_{x,t}} \lesssim \|\varphi\|_{X^{s,1/2+}} \|\varphi\|^2_{X^{\frac{d-2}{2}+2\varepsilon+,1/2+}} \lesssim \|\varphi\|^3_{X^{s,1/2+}},$$

for $s > \frac{d-2}{2}$ and $\varepsilon < \frac{1}{2}(s - \frac{d-2}{2})$.

The bound (44) will follow from several applications of Cauchy-Schwarz. We assume $n_1 \geq n_2, n_2$, so that the summation restriction of $\Lambda_2(n)$ implies that $\langle n_2\rangle\langle n_3\rangle \ll n^{1/2}$, hence

$$\frac{1}{(\langle n_2\rangle\langle n_3\rangle)^{s-\frac{d-2}{2}-} n^{1-\varepsilon}} \lesssim \frac{n^{0-}}{(\langle n_2\rangle\langle n_3\rangle)^{1/2+}}$$

for $\varepsilon < \frac{1}{2}\min(s - \frac{d-5}{2}, 2)$. Now, given $s > \frac{d-2}{2}$ and $\varepsilon < \frac{1}{2}\min(s - \frac{d-5}{2}, 2)$ we use Lemma 4.6 to bound κ by $O((n_2n_3)^{\frac{d-2}{2}+})$ and invoke the above display to find

$$\|B(v)\|_{C^0_t H^{s+\varepsilon}_x} \lesssim \| \sum_{\substack{n_1,n_2,n_3 \\ n\neq n_1, \Lambda_2(n)}} |\widehat{v}_{n_1} \widehat{\bar{v}}_{n_2} \widehat{v}_{n_3}| \frac{\langle n\rangle^{s+\varepsilon}(n_2n_3)^{\frac{d-2}{2}+}}{n|n-n_1|} \|_{C^0_t \ell^2_n}$$
$$\lesssim \| \sum_{\substack{n_1,n_2,n_3 \\ n\neq n_1, \Lambda_2(n)}} |\widehat{v}_{n_1} \widehat{\bar{v}}_{n_2} \widehat{v}_{n_3}| \frac{\langle n_1\rangle^{s-}(\langle n_2\rangle\langle n_3\rangle)^{s-\frac{1}{2}-}}{|n-n_1|} \|_{C^0_t \ell^2_n}$$
$$\lesssim \|u\|^2_{C^0_t H^s_x} \| \sum_{n\neq n_1} \frac{\langle n_1\rangle^{0-}|\widehat{u}_{n_1}|}{|n-n_1|} \|_{C^0_t \ell^2_n} \lesssim \|u\|^3_{C^0_t H^s_x},$$

by Young's and Cauchy-Schwarz. □

Lemma 4.12. *Let $d \geq 2$, $s > \frac{d-2}{2}$, and $0 \leq \varepsilon < \min(s - \frac{d-2}{2}, 1)$, then*

$$\|N_{0,1}(v)\|_{X_T^{s+\varepsilon,-1/2+}} \lesssim_\varepsilon \|v\|^3_{X_T^{s,1/2+}}.$$

Proof. We define

$$\mathcal{F}_{x,t}(\varphi)(n,\tau) := |\mathcal{F}_{x,t}(v)(n,\tau)|,$$
$$\tilde{\kappa}(n,n,n_2,n_3) := \kappa(n,n,n_2,n_3) - \frac{1}{\pi\omega_d}\int_0^\pi Y_{n_2}(\theta)Y_{n_3}(\theta)\,d\theta,$$

so that we are reduced to bounding

$$\begin{aligned}
&|\mathcal{F}_{x,t}(N_{0,1}(v))(n,\tau)| \\
&\quad\lesssim \int \mathcal{F}_{x,t}(\varphi)(n,\tau_1) \sum_{n_2,n_3\leq n} \mathcal{F}_{x,t}(\bar{\varphi})(n_2,\tau_2)\mathcal{F}_{x,t}(\varphi)(n_3,\tau-\tau_1-\tau_2)|\tilde{\kappa}(n,n,n_2,n_3)|\,d\tau_1 d\tau_2 \\
&+ \int \mathcal{F}_{x,t}(\varphi)(n,\tau_1) \sum_{\max(n_2,n_3)>n} \mathcal{F}_{x,t}(\bar{\varphi})(n_2,\tau_2)\mathcal{F}_{x,t}(\varphi)(n_3,\tau-\tau_1-\tau_2)|\tilde{\kappa}(n,n,n_2,n_3)|\,d\tau_1 d\tau_2 \\
&\qquad := \mathcal{F}_{x,t}(N_{0,1}^\ell + N_{0,1}^h).
\end{aligned}$$

In order to handle $N_{0,1}^\ell$ we see that by Lemma 4.6 that we have

$$\kappa(n,n,n_2,n_3) - \frac{1}{\pi}\int_0^\pi Y_{n_2}(\theta)Y_{n_3}(\theta)\,d\theta = O\left(\frac{(n_2n_3)^{\frac{d-1}{2}+}}{n}\right),$$

and hence it suffices to bound

$$\int \mathcal{F}_{x,t}(\varphi)(n,\tau_1) \sum_{n_2,n_3\leq n} \mathcal{F}_{x,t}(\bar{\varphi})(n_2,\tau_2)\mathcal{F}_{x,t}(\varphi)(n_3,\tau-\tau_1-\tau_3)\frac{(n_2n_3)^{\frac{d-1}{2}+}}{n}\,d\tau_1 d\tau_2, \tag{54}$$

in $X^{s,1/2+}$.

Noticing that

$$|H_n(n,n_2,n_3,n)| = |(n_2+n_3+d-1)(n_2-n_3)| \gtrsim \max(n_2,n_3)|n_2-n_3|,$$

we separate out two cases:

I) $n_2 = n_3$

II) $|H_n| \gtrsim \max(n_2,n_3)|n_2-n_3|$.

In the first case, we see that the $X^{s,1/2+}$ norm of (54) reduces to bounding

$$\begin{aligned}
\|(54)\|_{X^{s,1/2+}} &\lesssim \left\| \langle n\rangle^{s+\varepsilon}\mathcal{F}_x(\varphi)(n,t)\sum_{n_2\leq n}|\mathcal{F}_x(\bar{\varphi})(n_2,t)|^2\frac{n_2^{d-1+}}{n}\right\|_{L_t^2\ell_n^2} \\
&\lesssim \|\varphi\|_{L_t^\infty H_x^s}\|\varphi\|_{L_t^\infty H_x^{\frac{d-2+\varepsilon}{2}}}\|\varphi\|_{L_t^2 H_x^{\frac{d-2+\varepsilon}{2}}} \lesssim \|\varphi\|^3_{X_T^{s,1/2+}},
\end{aligned}$$

for $\varepsilon < \min\left(s - \frac{d-2}{2}, 1\right)$.

We now assume that we have modulation considerations at play. Specifically, if we are in case II then we again find that

$$\langle \tau_2 + n_2(n_2+d-1)\rangle + \langle \tau_3 - n_3(n_3+d-1)\rangle \gtrsim |H_n| \gtrsim \max(n_2,n_3)|n_2-n_3|,$$

and hence we may assume that $\langle \tau_2 + n_2^2\rangle \gtrsim \max(n_2,n_3)|n_2-n_3|$. It follows that

$$\frac{\langle n\rangle^{\varepsilon}(n_2n_3)^{\frac{d-1}{2}+}}{n} \lesssim \frac{\langle \tau_2 + n_2(n_2+d-1)\rangle^{1/2+}(n_2n_3)^{s-1/4-}}{|n_2-n_3|^{1/2+}},$$

for $\varepsilon < \min(s-\frac{d-2}{2},1)$ and $s > \frac{d-2}{2}$, so that we find the contribution of the above to (54) satisfies

$$\begin{aligned}\|(54)\|_{X^{s,1/2+}} &\lesssim \|\varphi\|_{L^\infty_t H^s_x} \\ &\times \Big\| \int \sum_{n_2,n_3\le n} \tfrac{\langle \tau_2+n_2(n_2+d-1)\rangle^{1/2+}(n_2n_3)^{s-1/4+}}{|n_2-n_3|^{1/2+}} \mathcal{F}_{x,t}(\bar\varphi)(n_2,\tau_2)\mathcal{F}_{x,t}(\varphi)(n_3,\tau-\tau_2)\,d\tau_2\Big\|_{L^2_\tau \ell^\infty_n} \\ &\lesssim \|\varphi\|^3_{X^{s,1/2+}},\end{aligned}$$

by Proposition 4.8 with $\delta = 1/2+$. We note that the proof in the case that $\langle \tau_2 + n_2^2\rangle \ll \max(n_2,n_3)|n_2-n_3|$ follows as above, with the only difference being which term is placed in L^2_t. This provides smoothing of order

$$0 \le \varepsilon < 2\min(s-\tfrac{d-2}{2},1/2), \text{ for } s > \tfrac{d-2}{2}.$$

We now assume that $\max(n_2,n_3) \gtrsim n$, so as to handle the contribution of $N^h_{0,1}$. By positivity and (35) we observe

$$\tilde\kappa(n,n,n_2,n_3) \lesssim \kappa(n,n,n_2,n_3) + O((n_2n_3)^{\frac{d-2}{2}+}) \tag{55}$$

and

$$\begin{aligned}\langle n\rangle^{\varepsilon} &\sum_{\max(n_2,n_3)\ge n} \mathcal{F}_{x,t}(\bar\varphi)(n_2,\tau_2)\mathcal{F}_{x,t}(\varphi)(n_3,\tau_3) \\ &\lesssim \sum_{\max(n_2,n_3)\ge n} \max(n_2,n_3)^{\varepsilon}\mathcal{F}_{x,t}(\bar\varphi)(n_2,\tau_2)\mathcal{F}_{x,t}(\varphi)(n_3,\tau_3),\end{aligned}$$

so that the contribution to $N^h_{0,1}$ corresponding to the first term of (55) may be handled using the bilinear L^2 Strichartz estimate 4.10. That is, the contribution of the above satisfies (by duality and the bilinear L^2 estimate)

$$\|N^h_{0,1}\|_{X^{s+\varepsilon,1/2+}} \lesssim \|\varphi J^s\varphi J^\varepsilon\varphi\|_{X^{0,1/2+}} \lesssim \|\varphi\|_{X^{s,1/2+}}\|\varphi\|^2_{X^{\frac{d-2}{2}+\varepsilon,1/2+}} \lesssim \|\varphi\|^3_{X^{s,1/2+}},$$

given $0 \le \varepsilon < s - \frac{d-2}{2}$ and $s > \frac{d-2}{2}$.

As for the contribution of the second term of (55), we observe that

$$\langle n\rangle^{\varepsilon} \sum_{\max(n_2,n_3)\geq n} \mathcal{F}_{x,t}(\bar{\varphi})(n_2,\tau_2)\mathcal{F}_{x,t}(\varphi)(n_3,\tau_3)(n_2n_3)^{\frac{d-2}{2}+}$$
$$\lesssim \sum_{\max(n_2,n_3)\geq n} \mathcal{F}_{x,t}(\bar{\varphi})(n_2,\tau_2)\mathcal{F}_{x,t}(\varphi)(n_3,\tau_3)(n_2n_3)^{\frac{d-2}{2}+\varepsilon+}.$$

We may then bound the contribution to $N^h_{0,1}$ by using the exact same case work as that done to bound $N^{\ell}_{0,1}$. This yields smoothing of $0 \leq \varepsilon < s - \frac{d-2}{2}$ for $s > \frac{d-2}{2}$. □

Lemma 4.13. *Let $d \geq 2$, $s > \frac{d-2}{2}$, and $0 \leq \varepsilon \leq s - \frac{d-2}{2}$, then*

$$\|N_{0,2}(v)\|_{X_T^{s+\varepsilon,-1/2+}} \lesssim_{\varepsilon} \|v\|^3_{X_T^{s,1/2+}}.$$

Proof. This proof follows in exactly the same way as the proof of the prior lemma in the situation $\max(n_2,n_3) \gtrsim n$. In particular, if $\mathcal{F}_{x,t}(\varphi) = |\mathcal{F}_{x,t}(v)|$ then we find

$$\langle n\rangle^{s+\varepsilon}\mathcal{F}(\varphi)(n)^2 \sum_{n_2} \mathcal{F}(\bar{\varphi})(n_2)\kappa(n,n,n_2,n) \tag{56}$$
$$= \langle n\rangle^{s}\mathcal{F}(\varphi)(n)\langle n\rangle^{\varepsilon}\mathcal{F}(\varphi)(n) \sum_{n_2} \mathcal{F}(\bar{\varphi})(n_2)\kappa(n,n,n_2,n),$$

and hence by the bilinear L^2 estimate and duality with $w \in X^{0,1/2-}$:

$$\|(56)\|_{X^{0,1/2+}} \lesssim \|wJ^{\varepsilon}_x\varphi\|_{L^2_{x,t}}\|\varphi J^s\varphi\|_{L^2_{x,t}} \lesssim \|\varphi\|^3_{X^{s,1/2+}},$$

for $\varepsilon < s - \frac{d-2}{2}$ and $s > \frac{d-2}{2}$. □

5. Appendix: Strichartz Estimates and the Cubic NLS

In this appendix we establish a slightly strengthened bilinear Strichartz estimate for a class of functions in $\mathbb{S}^2$, which is useful for lower bounds on the fractal dimension of the graph of the free solution. As a corollary we establish an improved well-posedness statement for functions that are supported on the zonal harmonics that matches the statement on $\mathbb{T}^2$.

We recall the space $\mathcal{Z}^s$ and define $\mathcal{B}^s$ as

$$\mathcal{Z}^s(\mathbb{S}^2) := \left\{ f \in H^s(\mathbb{S}^2) : f(\theta,\phi) = \sum_n a_n Y_n(\theta,\phi) \right\},$$
$$\mathcal{B}^s(\mathbb{S}^2) := \left\{ f \in H^s(\mathbb{S}^2) : f(\theta,\phi) = \sum_n \sum_{j\in\{\pm 1\}} a_{nj} Y_n^{jn}(\theta,\phi) \right\}.$$

The first of these spaces coincides with the space of functions in H^s that are supported only on the zonal harmonics, whereas the second corresponds to the space of functions supported only on the gaussian beams $Y_n^{\pm n}$.

Lemma 5.1. *Suppose that $f, g \in \mathcal{Z}^s$. Then for $N \geq M$ dyadic and every $\varepsilon > 0$ we have*

$$\|P_N(e^{it\triangle_{\mathbb{S}^2}} f)P_M(e^{it\triangle_{\mathbb{S}^2}} g)\|_{L^2_{x,t\in[0,2\pi]}} \lesssim M^\varepsilon \|P_N(f)\|_{L^2_x} \|P_M(g)\|_{L^2_x}. \tag{57}$$

Proof. We first assume that $f, g \in \mathcal{Z}^s$. Represent

$$f = \sum_n f_n Y_n,$$

and similarly for g. Following [BGZ], we find by Parseval's (for $t \in [0, 2\pi]$)

$$\begin{aligned} \|P_N(e^{it\triangle_{\mathbb{S}^2}} f)P_M(e^{it\triangle_{\mathbb{S}^2}} g)\|^2_{L^2_{x,t}} &= \sum_{\tau=0}^{\infty} \Bigg\| \sum_{\substack{\tau=n(n+1)+m(m+1)\\ n\sim N,\, m\sim M}} f_n g_m Y_n Y_m \Bigg\|^2_{L^2_x} \\ &\leq \sum_{\tau=0}^{\infty} \alpha_{NM}(\tau) \sum_{\substack{\tau=n(n+1)+m(m+1)\\ n\sim N,\, m\sim M}} |f_n g_m|^2 \|Y_n Y_m\|^2_{L^2_x}, \end{aligned} \tag{58}$$

where

$$\alpha_{NM}(\tau) = \# \left\{ \substack{(n,m)\in\mathbb{N}^2,\, n\sim N,\, m\sim M \\ \tau=n(n+1)+m(m+1)} \right\}. \tag{59}$$

By the divisor bound, we have that $\sup_\tau \alpha_{NM}(\tau) \lesssim M^\varepsilon$ for any $\varepsilon > 0$.

It then suffices to estimate the quantity $Y_n Y_m$ in L^2. By the first bound in (35) we have

$$\|Y_n Y_m\|^2_{L^2(\mathbb{S}^2)} \lesssim \int_0^{\pi/2} \frac{\theta n m}{\langle n\theta\rangle\langle m\theta\rangle} d\theta \leq \int_0^{\pi/2} \frac{m}{\langle m\theta\rangle} d\theta \lesssim M^\epsilon.$$

Combining this with the bound for $\alpha_{NM}(\tau)$ and summing in τ we find

$$(58) \lesssim M^\varepsilon \|P_N(f)\|^2_{L^2_x} \|P_M(g)\|^2_{L^2_x}.$$

For more details, see [BGZ]. □

Remark 5.2. *The above calculation isn't generic. That is, $f \in \mathcal{B}^s$ of the form $f = Y_n^n$ saturates the L^4 inequality on $\mathbb{S}^2$. Indeed,*

$$Y_n^n = \frac{(-1)^n}{2^n n!} \sqrt{\frac{(2n+1)!}{4\pi}} \sin^n \theta e^{in\phi},$$

so

$$\|f\|^4_{L^4_{x,t}} \sim n \int_0^\pi \sin^{4n+1}\theta \, d\theta \sim n \frac{2^n (n!)^2}{(2n+1)!} \sim \sqrt{n},$$

by Stirling's approximation.

A similar calculation can be done to show that the Zonal harmonics essentially saturate the L^4 inequality for $d \geq 3$, [BGZ2].

As a corollary of the above lemma, we find local well-posedness for $s > 0$ for functions on $\mathbb{S}^2$ that are independent of ϕ.

Corollary 5.3. *Let $s > 0$ and consider the space $\mathcal{Z}^s(\mathbb{S}^2) \subset H^s(\mathbb{S}^2)$. Then the equation*

$$\begin{cases} i\partial_t u + \triangle u \pm |u|^2 u = 0 \\ u(x,0) = f(x) \in \mathcal{Z}^s(\mathbb{S}^2) \end{cases} \tag{60}$$

is locally well-posed for any $s > 0$.

Remark 5.4. *In light of Remark 5.2 we see that the generic statement is $s > 1/4$ and cannot, in general, be improved.*

6. Appendix: The Torus Case

Before proceeding with theorem statements, we comment that Theorem 2.4 and Theorem 2.8 hold as stated for $\mathbb{T}^d$, and their proofs are standard. With preliminaries out of the way, we can prove a theorem analogous to Theorem 1.1 relatively easily for the Torus. In particular, we obtain the following natural generalization of the one dimensional statement.

Theorem 6.1. *Let $f : \mathbb{T}^d \to \mathbb{R}$ and define $f_N(x) := \sum_{N < \max_i\{|m_i|\} \leq 2N} \widehat{f}(m) e^{im\cdot x}$. Assume f satisfies $\|f_N\|_{L^1_x} \lesssim N^{-(\frac{d}{2}+s)}$ for some $s \in (0,1]$. Define*

$$u(x,t) := \sum_{m\in\mathbb{Z}^d} e^{i|m|^2 t} \widehat{f}(m) e^{im\cdot x},$$

and

$$H_N(x,t) := \sum_{N < \max_i\{|m_i|\} \leq 2N} e^{it|m|^2} e^{im\cdot x}.$$

Then for almost all t, $u(t,x) \in C^{s-}$ and $\dim_t(f) \leq (d+1) - s$.

Before moving on to the proof of the theorem, we will need the following proposition that easily follows from factorization and the 1 dimensional Weyl bound.

Proposition 6.2. *For $N \geq 1$ dyadic, and almost every t:*

$$\sup_{x\in\mathbb{T}^d} \left| \sum_{N \leq \max\{|m_1|,\cdots,|m_d|\} < 2N} e^{it|m|^2} e^{im\cdot x} \right| \lesssim N^{\frac{d}{2}+}.$$

Theorem 6.1. This proof is substantially easier than Theorem 1.1. We write

$$\|P_N(u(\cdot,t)\|_{L^\infty_x} = \|f_N * H_N\|_{L^\infty_x} \lesssim \|f_N\|_{L^1_x} \|H_N(\cdot,t)\|_{L^\infty_x} \lesssim N^{\frac{d}{2}-\left(\frac{d}{2}+s\right)} = N^{-s},$$

for almost every t. It follows by Theorem 2.4 that for almost every t we have the fractal dimension of the graph of u is bounded above by $(d+1) - s$. □

Remark 6.3. *The L^1 condition assumed above is rather unwieldy and is way too strong to obtain a lower bound. We correct this in the following two theorems, which are more analogous to the one dimensional statements for BV functions.*

We now, for simplicity of statement, restrict ourselves to $d = 2$. It may be desirable to estimate the dimension of the graph under an assumption on the Fourier coefficients, in which case we will need more information about how the Fourier coefficients of f behave. In particular, we'll need to define, for $m = (m_1, m_2)$:

$$\sigma_m(f) = \widehat{f}(m_1 + 1, m_2 + 1) - \widehat{f}(m_1 + 1, m_2) - \widehat{f}(m_1, m_2 + 1) + \widehat{f}(m_1, m_2),$$

as well as

$$\sigma_m^1(f) = \widehat{f}(m_1 + 1, m_2) - \widehat{f}(m_1, m_2)$$
$$\sigma_m^2(f) = \widehat{f}(m_1, m_2 + 1) - \widehat{f}(m_1, m_2).$$

These terms measure how much $\widehat{f}$ varies near the point $m \in \mathbb{Z}^2$.

With the prior definition out of the way, we find Theorem 6.4 by a direct application of summation-by-parts.

Theorem 6.4. *Let $m \in \mathbb{Z}^2$, $s \in (0, 1)$, and suppose that the Fourier coefficients of f satisfy, for $1 \le i \le 2$:*

$$|\widehat{f}(m)| \lesssim \frac{1}{\langle m \rangle^{1+s}}, \ |\sigma_m^i(f)| \lesssim \frac{1}{\langle m \rangle^{2+s}}, \ |\sigma_m(f)| \lesssim \frac{1}{\langle m \rangle^{3+s}}. \tag{61}$$

Let

$$u(x, t) := \sum_{m \in \mathbb{Z}^d} e^{i|m|^2 t} \widehat{f}(m) e^{im \cdot x},$$

be the solution emanating from f.

i) Then for almost all t: $u(x, t) \in C^{(s-1)-}$ and hence $\dim_t(f) \le 3 - s$.
ii) If $u(x, t)$ is continuous in x and $r_0 = sup_r\{r : f \in H^r\}$, then

$$\dim_t(f) \ge 3 + s - 2r_0.$$

In particular, if f satisfies (61) with $s = \frac{1}{2}$ and $f \notin H^{1/2}$, then for almost every time, t, we have

$$\dim_t(f) = \tfrac{5}{2}.$$

Proof. The first claim follows from a direct application of summation-by-parts (done twice) to the function

$$u_N(x, t) = \sum_{N < \max_i\{|m_i|\} \le 2N} e^{i|m|^2 t} e^{im \cdot x} \widehat{f}(m).$$

The second follows by the Strichartz estimate (17) and the standard interpolation argument. □

6.1. **Bounded Variation on the d-Torus.** In this subsection we comment on and provide some results relating to higher dimensional generalizations to bounded variation and their applications to estimations of the dimension of the graph of the free solution associated to (1).

For $1 \leq i \leq d$ and $\lambda_i \in \mathbb{N}$ we choose $\{x_j^{(i)}\}_{j=1}^{\lambda_i}$ so that

$$0 = x_1^{(i)} < \cdots < x_{\lambda_i}^{(i)} = 1.$$

We define Π to be the collection of all tuples of d such sequences.

With these sequences, we define the difference operators Δ_{ij} for $1 \leq j \leq \lambda_i$ to be

$$\Delta_{ij} f := f(y_1, \cdots, y_{i-1}, x_{j+1}^{(i)}, y_{i+1}, \cdots, y_d) - f(y_1, \cdots, y_{i-1}, x_j^{(i)}, y_{i+1}, \cdots, y_d).$$

Definition 6.5 (Vitali Bounded Variation)**.** *Let V be the space of functions f satisfying*

$$\sup_{\Pi} \sum_{1 \leq j \leq \lambda_i - 1} \left| (\prod_i \Delta_{ij}) f \right| < \infty.$$

There is also a slight modification on this space, Fréchet Bounded Variation.

Definition 6.6 (Fréchet Bounded Variation)**.** *Let $\epsilon_i, \nu_j \in \{-1, 1\}$. Then let F be the space of functions satisfying*

$$\sup_{\Pi, \epsilon_{ij}} \left| \sum_{1 \leq j \leq \lambda_i - 1} (\prod_i \epsilon_i \nu_j \Delta_{ij}) f \right| < \infty.$$

It's clear that $V \subset F$, but these two spaces do not coincide, [ClAd]. The main benefit of these spaces is that if $f \in V(\mathbb{T}^d)$ and g is continuous then, we have a generalized Stieltjes integration-by-parts formula of the form

$$\int_{\mathbb{T}^d} f \, dg = (-1)^d \int_{\mathbb{T}^d} g \, df, \tag{62}$$

where dg can morally be thought of as $\partial_{x_1, \cdots, x_d} g$. Similarly, when $g(x_1, \cdots, x_d) = \prod_i \eta_i(x_i)$ for continuous η_i then (see, for example [Cl]) for $f \in F(\mathbb{T}^d)$ we again have the formula (62).

It's interesting to note that, as a consequence of these formulas, the Fourier coefficients of $f \in F(\mathbb{T}^d)$ satisfy

$$|\widehat{f}(m)| \lesssim \frac{1}{\prod_{\substack{1 \leq i \leq d \\ m_i \neq 0}} |m_i|},$$

which is again analogous to the one dimensional case.

With these definitions we can then show the following theorem.

Theorem 6.7. *Let $f \in V(\mathbb{T}^d) \backslash H^{1/2+}(\mathbb{T}^d)$ (or $F \backslash H^{1/2+}$). Then for almost every t, $\dim_t(f) = d + 1/2$.*

Proof. We prove this for $f \in V$, but the statement also holds for $f \in F$ by the factorization of the convolution kernel. Let

$$\widetilde{H}_N(x,t) = \sum_{N \leq \max_i(|m_i|) < 2N} \frac{e^{it|m|^2 + ix\cdot m}}{R(m)}, \text{ where}$$

$$R(m) = \prod_{\substack{i=1 \\ m_i \neq 0}}^{d} m_i.$$

It follows that $\widetilde{H}_N(x,t)$ is continuous in x for almost every t, and hence for almost every t we find by (62):

$$\int f(y) H_N(x-y)\, dx = (-1)^d \int \tilde{H}_N(x-y)\, df(y),$$

so that

$$\|P_N(u)\|_{L^\infty} \lesssim \|\widetilde{H}_N(\cdot,t)\|_{L^\infty} |df|(\mathbb{T}^d) \lesssim_f N^{-\frac{1}{2}+},$$

where we have again invoked 6.2 and summation-by-parts. It follows that $\dim_t(f) \leq (d+1) - 1/2 = d + 1/2$.

For the lower bound, we see that

$$\|P_N(u)\|_{L^2} \lesssim \|\widetilde{H}_N(\cdot,t)\|_{L^2} \lesssim N^{-\frac{1}{2}+},$$

so that we find the appropriate level for u is $H^{1/2+}$, motivating the statement of the theorem. Now, noting that u is continuous, we assume that $u \notin H^{1/2+}$ and interpolate to obtain that $\dim_t(f) \geq (d+1) - 1/2 = d + 1/2$. □

It's worth noting that the above is not in any sense optimal. Indeed, consider a small cube, R, supported in $[0,1]^d$. If the sides are parallel to the coordinate axes then the characteristic function of R will clearly be in V (and hence F), but if we slightly rotate the square R, then it immediately leaves both classes.

We can say more, however: the characteristic function for any simplex $P \subset \mathbb{T}^2$ has fractal dimension that is exactly $\frac{5}{2}$.

Theorem 6.8. *Let P be a polygon in $\mathbb{T}^2$, and χ_P the characteristic function associated to P. Then $\dim_t(f) = \frac{5}{2}$ for almost all t.*

Proof. By triangulation we reduce the problem to considering triangles. By utilizing subtraction we can further reduce the problem to considering right triangles with two sides parallel to the coordinate axis. Without loss of generality we can then just consider a non-degenerate right triangle, $\mathcal{T}$, with vertices $(0,0)$, $(x_2,0)$, and (x_2,y_2).

We readily calculate the nontrivial $(n,m\neq 0)$ Fourier coefficients of $\chi_{\mathcal{T}}$ to be

$$\widehat{\chi_{\mathcal{T}}}(n,m)=\begin{cases}\frac{y_2(e^{2\pi i n x_2}-1)e^{-2\pi i n y_2}}{4\pi^2n^2} & m=-\frac{nx_2}{y_2},\ n\neq 0\\ \frac{y_2(e^{-2\pi i(x_2,y_2)\cdot(n,m)}-1)}{4\pi^2n(my_2+nx_2)}+\frac{(e^{2\pi i m y_2}-1)e^{-2\pi i y_2(m+n)}}{4\pi^2nm} & \text{else.}\end{cases}\tag{63}$$

The ommitted cases are all trivial by one dimensional theory, and so is the second term in the last case. We thus restrict ourselves to considering the term (modulo constants)

$$\frac{e^{-i(x_2,y_2)\cdot(n,m)}}{n(m+n)},$$

which will be sufficient to prove the full result by noting that the only difference in the argument will be nearest integer considerations.

We first let $u=e^{-it\triangle}\chi_{\mathcal{T}}$ and consider when $|m|<|n|$, and note that in order to show the upper bound we need to bound the L^∞ norm of

$$\begin{aligned}P_N(u)&=\sum_{N<|n|\leq 2N}\frac{e^{itn^2+i(x-x_2)n}}{n}\sum_{-|n|<m<|n|}\frac{e^{itm^2+i(y-y_2)m}}{n+m}\\&=\sum_{N<|n|\leq 2N}\frac{e^{2itn^2+i(x-x_2)n-(y-y_2)|n|}}{n}\sum_{0<h<2|n|}\frac{e^{ith^2+i(y-y_2-2|n|)h}}{h}\\&=\sum_{0<h<4N}\frac{e^{ith^2+i(y-y_2-2|n|)h}}{h}\sum_{\max\left(N,\frac{h+2N}{3}\right)<|n|\leq 2N}\frac{e^{2itn^2+i(x-x_2)n-(y-y_2)|n|}}{n}.\end{aligned}$$

From this we find that

$$\|P_N(u)\|_{L^\infty}\lesssim\frac{N^{\frac{1}{2}+}\log N}{N}=O(N^{-1/2+}),$$

for almost every t uniformly in N,x_2,y_2. The region $|m|\geq|n|$ follows in the exact same manner, and hence $u\in C^{1/2-}$ and $\dim_t(f)\leq\frac{5}{2}$.

As for the lower bound, we note that it's a standard result on $\mathbb{R}^d$ that the characteristic function of a measurable set with positive measure are not in $H^{1/2}$. These results trivially extend to $\mathbb{T}^d$, and hence we conclude $\chi_P\in H^{1/2-}\setminus H^{1/2}$ as well as the lower bound as before. $\square$

The above generalizes to polytopes in $[-1,1]^d$. Indeed, Stokes theorem can be used to write the Fourier transform of the characteristic function of a d-dimensional polytope as a sum of products, each of which contains d homogeneous algebraic factors of degree -1, see Theorem 1 of [DLR]. It then follows using the exact same change of variables as above that the following theorem holds.

Theorem 6.9. *Let P be a polytope in $\mathbb{T}^d$, and χ_P the characteristic function associated to P. Then for almost every t, $\dim_t(\chi_P) = d + \frac{1}{2}$.*

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